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STABILITY THEORY OF MULTISTEP METHODS

by

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ABSTRACT

The numerical solution of differential equations of the form $y' = f(x, y)$ using predictor-corrector multistep methods is examined with particular emphasis on the stability concepts. Computational methods for determining the region of stability for single multistep methods and predictor-corrector pairs are expounded, and two subroutines have been written to compute the boundary of the region of stability for the single multistep methods and predictor-corrector pairs.

STABILITY THEORY OF MULTISTEP METHODS

By S. M. Keathley and T. J. Aird*
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SUMMARY

The numerical solution of differential equations of the form $y' = f(x, y)$ using predictor-corrector multistep methods is examined with particular emphasis on the stability concepts. Computational methods for determining the region of stability for single multistep methods and predictor-corrector pairs are expounded. Two subroutines have been written to compute the boundary of the region of stability for the single multistep methods and predictor-corrector pairs.

INTRODUCTION

The results of recent studies concerning predictor-corrector multistep methods for numerically solving differential equations of the form $y' = f(x, y)$ are investigated with particular emphasis on the stability concepts, including justification of the theory for systems of differential equations. Determination of the region of stability for single multistep methods and predictor-corrector pairs by computational methods is presented as outlined by Crane and Klopfenstein (ref. 1) and Krogh (ref. 2).

A FORTRAN IV subroutine STBL1 has been written to compute the boundary of the region of stability for single multistep methods of either the predictor or corrector type, and is described along with examples of the Adams-Bashforth predictors and the Adams-Moulton correctors. A FORTRAN IV subroutine STBL2 has been written to compute the boundary of the region of stability for predictor-corrector multistep methods for which the corrector is applied only once and is described along with examples of the Bashforth-Moulton pairs.

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SYMBOLS

$f_{n+\ell} = f\left(x_{n+\ell}, y_{n+\ell}\right)$	
h	step size
$\bar{h} = \lambda h$	
$x_{n+\ell} = x_n + \ell h$	
$y_{n+\ell} = y\left(x_n + \ell h\right)$	approximate solution to $y' = f(x, y)$ at $x = x_{n+\ell}$
$\hat{y}_\ell = \hat{y}\left(x_\ell\right)$	solution to $y' = f(x, y)$ at $x = x_\ell$
$\hat{y}'_\ell = f\left(x_\ell, \hat{y}_\ell\right)$	
$\left(\alpha_\ell, \beta_\ell\right), \left(\gamma_\ell, \delta_\ell\right)$	coefficients for a multistep method
$\epsilon_\ell = y_\ell - \hat{y}_\ell$	
$\epsilon'_\ell = y'_\ell - \hat{y}'_\ell$	
$\lambda = \frac{\partial f}{\partial y}$	
$0\left(h^\ell\right)$	terms of order h^ℓ

DEVELOPMENT OF THEORY

Multistep Methods

A multistep method for solving the initial value problem $y' = f(x, y)$, $y\left(x_0\right) = y_0$ can be represented by the following differential-difference equation:

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_0 y_n = h\left(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \ldots + \beta_0 f_n\right) \quad (1)$$

where h is the step size, $x_n = x_0 + nh$, $y_{n+j} = y\left(x_{n+j}\right)$, and $f_{n+j} = f\left(x_{n+j}, y_{n+j}\right)$.

In order that multistep methods be useful in the numerical solution of differential equations, it is necessary that the value of the expression

$$L\left[\hat{y}(x)\right] = \sum_{\ell=0}^{k} \alpha_\ell \hat{y}_{n+\ell} - h \sum_{\ell=0}^{k} \beta_\ell \hat{y}'_{n+\ell} \tag{2}$$

be small when h is small (ref. 3). By expanding the terms of $L\left[\hat{y}(x)\right]$ in a Taylor series about x in powers of h,

$$L\left[\hat{y}(x)\right] = c_0\hat{y}(x) + c_1 h\hat{y}'(x) + \ldots + c_n h^n \hat{y}^{(n)}(x) + \ldots \tag{3}$$

where

$$c_0 = \alpha_0 + \alpha_1 + \ldots + \alpha_k$$

$$c_1 = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k - \left(\beta_0 + \beta_1 + \ldots + \beta_k\right)$$

$$c_d = \frac{1}{d!}\left(\alpha_1 + 2^d\alpha_2 + \ldots + k^d\alpha_k\right) - \frac{1}{(d-1)!}\left(\beta_1 + 2^{d-1}\beta_2 + \ldots + k^{d-1}\beta_k\right)$$

For $d = 2, 3, \ldots$, it can be seen that

$$L\left[\hat{y}(x)\right] = 0\left(h^{d+1}\right) \tag{4}$$

if and only if $c_0 = c_1 = \ldots = c_d = 0$; but $c_{d+1} \neq 0$. If this is the case, then d will be called the degree of the multistep method, whereas the integer k will be called the order of the multistep method.

Example - The Adams-Bashforth $k = 1, d = 1$ method is

$$y_{n+1} - y_n = hf_n \tag{5}$$

and

$$L\left[\hat{y}(x_n)\right] = \hat{y}(x_n + h) - \hat{y}(x_n) \quad - h\hat{y}'(x_n)$$

$$\begin{aligned}
&= \hat{y}(x_n) \quad + h\hat{y}'(x_n) + \frac{h^2}{2}\hat{y}''(x_n) + \dots \\
&\quad -\hat{y}(x_n) \\
&\quad \qquad - h\hat{y}'(x_n) \\
\hline
&= 0 \qquad + 0 \qquad + \frac{h^2}{2}\hat{y}''(x_n) + \dots
\end{aligned} \tag{6}$$

so that $c_0 = c_1 = 0, \quad c_2 = \frac{1}{2}$.

The corresponding Adams-Moulton $k = 1, d = 1$ method is

$$y_{n+1} - y_n = hf_{n+1} \tag{7}$$

and

$$L\left[\hat{y}(x_n)\right] = \hat{y}(x_n + h) - \hat{y}(x_n) \quad - h\hat{y}'(x_n + h)$$

$$\begin{aligned}
&= \hat{y}(x_n) \quad + h\hat{y}'(x_n) + \frac{h^2}{2}\hat{y}''(x_n) + \dots \\
&\quad -\hat{y}(x_n) \\
&\quad \qquad - h\hat{y}'(x_n) - h^2\hat{y}''(x_n) \quad + \dots \\
\hline
&= 0 \qquad + 0 \qquad - \frac{h^2}{2}\hat{y}''(x_n) + \dots
\end{aligned} \tag{8}$$

so that $c_0 = c_1 = 0, \quad c_2 = -\frac{1}{2}$.

Stability for a Single Differential Equation

Consider the first order equation $y' = f(x, y)$ and let y_j, y'_j denote the solution computed by using the multistep method and $\hat{y}_j, \hat{y}'_j$ denote the exact solution for $j = n, \ldots, n + k$. Then, by substituting into the multistep formula first the computed values and then the exact values and subtracting, the following difference equation for the error is obtained.

$$\alpha_k \epsilon_{n+k} + \alpha_{k-1}\epsilon_{n+k-1} + \cdots + \alpha_0 \epsilon_n = h\left(\beta_k \epsilon'_{n+k} + \beta_{k-1}\epsilon'_{n+k-1} + \cdots + \beta_0 \epsilon'_n\right) \quad (9)$$

(The local truncation error $L[\hat{y}(x_n)]$ is assumed to be constant and is therefore neglected.)

where

$$\epsilon_j = y_j - \hat{y}_j$$

$$\epsilon'_j = y'_j - \hat{y}'_j$$

$$j = n, \ldots, n + k$$

Assuming that $\frac{\partial f}{\partial y} = \lambda$ is constant in the interval $\left(y_j, \hat{y}_j\right)$, it follows by the mean value theorem that $\epsilon'_j = \lambda \epsilon_j$, $j = n, \ldots, n + k$. Then the difference equation can be written as

$$\left(\alpha_k - \bar{h}\beta_k\right)\epsilon_{n+k} + \cdots + \left(\alpha_0 - \bar{h}\beta_0\right)\epsilon_n = 0 \quad (10)$$

where $\bar{h} = \lambda h$. This is a linear difference equation with constant coefficients whose solution may be found in the usual manner. First form the characteristic equation

$$\left(\alpha_k - \bar{h}\beta_k\right)z^k + \left(\alpha_{k-1} - \bar{h}\beta_{k-1}\right)z^{k-1} + \cdots + \left(\alpha_0 - \bar{h}\beta_0\right) = 0 \quad (11)$$

and let $\xi_1, \xi_2, \ldots, \xi_k$ be the k roots of this equation. Then the solution to the difference equation is

$$\epsilon_n = c_1\xi_1^n + c_2\xi_2^n + \ldots + c_k\xi_k^n \tag{12}$$

if all roots are distinct. In the case of multiple roots, let

$$\xi_{j+1} = \xi_{j+2} = \ldots = \xi_{j+m}$$

then

$$c_{j+1}\xi_{j+1}^n + \ldots + c_{j+m}\xi_{j+m}^n$$

is replaced by

$$\left(c_{j+1} + nc_{j+2} + \ldots + n^{m-1}c_{j+m}\right)\xi_{j+1}^n$$

where $c_1, \ldots, c_k$ are constants determined by the initial conditions. In either case (distinct or multiple roots) if $|\xi_\ell| > 1$ for some ℓ, then it is clear that ϵ_n will be large for large n; that is, the error introduced at some stage of the solution process will grow at each successive stage of the process until finally the solution becomes useless. On the other hand if $|\xi_\ell| < 1$ for $\ell = 1, 2, \ldots, k$, then the error ϵ_n will be small for large n. The above analysis leads to the following definition: Let S* be an arcwise connected region in the complex plane which has zero as one of its boundary points. Then S* will be called the region of stability for

$$\begin{aligned} L\left[y\left(x_n\right)\right] = {} & \alpha_k y_{n+k} + \alpha_{k-1}y_{n+k-1} + \ldots + \alpha_0 y_n \\ & - h\left(\beta_k y'_{n+k} + \beta_{k-1}y'_{n+k-1} + \ldots + \beta_0 y'_n\right) \end{aligned} \tag{13}$$

provided all roots of equation (11) have an absolute value of less than 1 for all $\bar{h} \epsilon S^*$. If S is any other region satisfying these conditions, then S is a subset of S^*.

Ideally, the region of stability should be the left half plane. However, in practice this is seldom the case. The Adams-Moulton $k = 1, d = 2$ method

$$y_{n+1} - y_n = \frac{h}{2}\left(y'_{n+1} + y'_n\right) \tag{14}$$

has this ideal region of stability since the characteristic equation

$$\left(1 - \frac{\bar{h}}{2}\right)z - \left(1 + \frac{\bar{h}}{2}\right) = 0$$

has root $z = \frac{2 + \bar{h}}{2 - \bar{h}}$ which has an absolute value of less than 1 when $Re(\bar{h}) < 0$. This concept of stability can best be understood by considering the first order linear differential equation $y' = \lambda y$ with initial condition $y(0) = 1$ where λ is a constant. If this equation is substituted into the multistep formula

$$\alpha_k y_{n+k} + \dots + \alpha_0 y_n = \bar{h}\left(\beta_k y_{n+k} + \dots + \beta_0 y_n\right) \tag{15}$$

and if $\xi_1, \xi_2, \dots, \xi_k$ are the roots of the characteristic equation, it would be desirable to have $\xi_1 \simeq \exp(h\lambda)$ and $\xi_2 = \dots = \xi_k = 0$. Then $|\xi_1| < 1$ when $Re(\lambda) < 0$, and $|\xi_1| \geq 1$ when $Re(\lambda) \geq 0$. That is to say, the multistep method has the left half plane as its region of stability.

Stability for a System of Differential Equations

Consider the system of differential equations $Y' = F(x, Y)$ where $Y = \left(y_1, y_2, \dots, y_n\right)$ and $F = \left(f_1, f_2, \dots, f_n\right)$. Let $Y\left(x_j\right)$, $Y'\left(x_j\right)$ denote the solution computed by using the multistep method and $\hat{Y}\left(x_j\right)$, $\hat{Y}'\left(x_j\right)$ denote the exact solution for $j = n, \dots, n + k$.

Then by substituting into the multistep formula first the computed values and then the exact values and subtracting, the following difference equation for the error is obtained

$$\alpha_k \epsilon(x_{n+k}) + \alpha_{k-1}\epsilon(x_{n+k-1}) + \ldots + \alpha_0 \epsilon(x_n) = h\left[\beta_k \epsilon'(x_{n+k}) + \beta_{k-1}\epsilon'(x_{n+k-1}) + \ldots + \beta_0 \epsilon'(x_n)\right] \tag{16}$$

(The local truncation error $L\left[\hat{Y}(x_n)\right]$ is assumed to be constant and is therefore neglected.)

where

$$\epsilon(x_j) = Y(x_j) - \hat{Y}(x_j)$$

$$\epsilon'(x_j) = Y'(x_j) - \hat{Y}'(x_j)$$

$$j = n, \ldots, n+k$$

Let $A = (a_{ij})$ where $a_{ij} = \frac{\partial f_i}{\partial y_j}$, the terms f_i and y_j being the ith and jth components of the vectors F and Y, respectively. Assuming that a_{ij} is constant, it follows by the mean value theorem that $\epsilon' = A\epsilon$. Then the difference equation can be written as

$$\alpha_k \epsilon(x_{n+k}) + \alpha_{k-1}\epsilon(x_{n+k-1}) + \ldots + \alpha_0 \epsilon(x_{n+k-1}) = hA\left[\beta_k \epsilon(x_{n+k}) + \beta_{k-1}\epsilon(x_{n+k-1}) + \ldots + \beta_0 \epsilon(x_n)\right] \tag{17}$$

Since every square matrix is similar to its Jordan canonical form, there exists a nonsingular matrix P such that $PAP^{-1} = J(A)$ where

$$J(A) = \begin{pmatrix} J_0 & 0 & 0 & . & . & . & 0 \\ 0 & J_1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & J_m \end{pmatrix} \tag{18}$$

is the Jordan canonical form of A and

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & \lambda_i \end{pmatrix} \tag{19}$$

where λ_i, $i = 1, \ldots, m$ are the eigenvalues of A and need not all be distinct. Then let $E = P\epsilon$, solve for $\epsilon = P^{-1}E$, substitute into the difference equation, and multiply both sides by P to obtain

$$\alpha_k E\left(x_{n+k}\right) + \ldots + \alpha_0 E\left(x_n\right) = hJ(A)\left[\beta_k E\left(x_{n+k}\right) + \ldots + \beta_0 E\left(x_n\right)\right] \tag{20}$$

The solution to each equation of this system of difference equations can be written in terms of powers of the roots of the characteristic equations

$$\left(\alpha_k - \lambda_i h\beta_k\right)z^k + \ldots + \left(\alpha_0 - \lambda_i h\beta_0\right) = 0; \quad i = 1, \ldots, m \tag{21}$$

For a more rigorous treatment see Lea (ref. 4).

Therefore, with a system of differential equations the stability is related to a set $\bar{h}_i = \lambda_i h$ where λ_i are the eigenvalues of the Jacobian matrix of the derivative functions. These eigenvalues may be complex numbers so that the stability must be considered in terms of complex values of $\bar{h}$.

Let ξ_ℓ, $\ell = 1, \ldots, k$, be the roots of the characteristic equation (21). Then if $\left|\xi_\ell\right| < 1$, $\ell = 1, \ldots, k$, the vector $E\left(x_n\right)$ will be small for large n and therefore the error vector $\epsilon\left(x_n\right) = P^{-1}E\left(x_n\right)$ will also be small.

The stability definition for a single equation will suffice for systems of equations.

Finding the Region of Stability

The problem may be restated as follows: For what values of $\bar{h}$ in the given characteristic equation

$$\left(\alpha_k - \bar{h}\beta_k\right) z^k + \dots + \left(\alpha_0 - \bar{h}\beta_0\right) = 0 \tag{22}$$

do the roots have an absolute value of less than 1. The direct method of finding the region of stability would be to vary $\bar{h}$ at each step, examining the roots to determine whether $\bar{h}$ was in the region. As k increases, the calculations involved soon become prohibitive. If the problem is looked at from a slightly different viewpoint, the answer is forthcoming. Because of the existing continuity between the roots and the coefficients of the characteristic equation, it is only necessary to calculate the values of $\bar{h}$ for which the equation has roots of an absolute value equal to 1. For this purpose it is necessary to rewrite the characteristic equation as

$$\sum_{\ell=0}^{k} \alpha_\ell z^\ell - \bar{h} \sum_{\ell=0}^{k} \beta_\ell z^\ell = 0 \tag{23}$$

which is linear in $\bar{h}$. Thus

$$\bar{h} = \frac{\sum_{\ell=0}^{k} \alpha_\ell z^\ell}{\sum_{\ell=0}^{k} \beta_\ell z^\ell} \tag{24}$$

To find those values of $\bar{h}$ for which the characteristic equation has roots of an absolute value equal to 1, let $z = e^{i\theta}$. Then

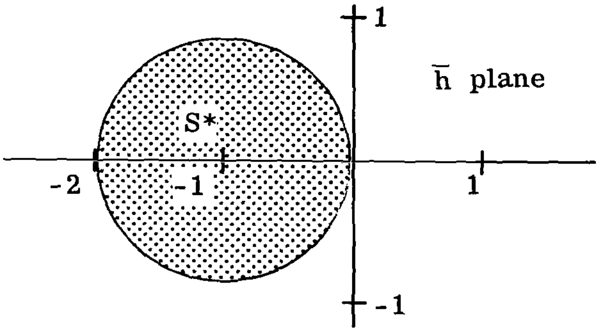
$$\bar{h}(\theta) = \frac{\sum_{\ell=0}^{k} \alpha_\ell e^{i\ell\theta}}{\sum_{\ell=0}^{k} \beta_\ell e^{i\ell\theta}} \tag{25}$$

By plotting $\bar{h}(\theta)$ for $0 \leq \theta \leq \pi$ and the conjugate, the region of stability is easily found. In any arcwise connected region, if the method is stable at one point, then it follows that it must be stable in the entire region; that is, $\bar{h}$ must cross the boundary $\bar{h}(\theta)$ in order to go from a stable region to an unstable region.

Example - The Adams-Bashforth $k = 1, d = 1$ method

$$y_{n+1} - y_n = hy'_n \tag{26}$$

shown in sketch a has the characteristic equation



Sketch a

$$z - (1 + \bar{h}) = 0 \tag{27}$$

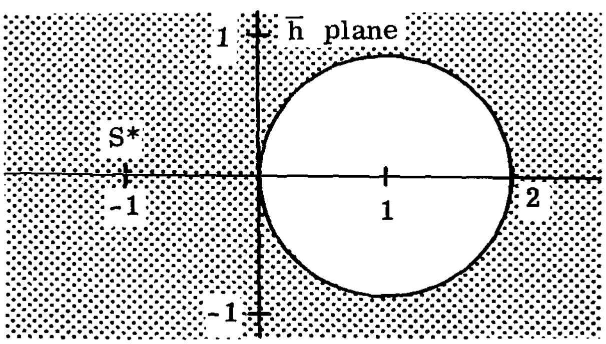
Therefore,

$$\bar{h}(\theta) = e^{i\theta} - 1 \tag{28}$$

The Adams-Moulton $k = 1, d = 1$ method

$$y_{n+1} - y_n = hy'_{n+1} \tag{29}$$

shown in sketch b has the characteristic equation



Sketch b

$$(1 - \bar{h})z - 1 = 0 \tag{30}$$

Therefore,

$$\bar{h}(\theta) = \frac{e^{i\theta} - 1}{e^{i\theta}} = 1 - e^{-i\theta} \tag{31}$$

A FORTRAN IV subroutine STBL1 has been written to compute the boundary of the region of stability for single multistep methods. The subroutine is described in appendix A along with examples of the Adams-Bashforth predictors and the Adams-Moulton correctors.

Stability of Predictor-Corrector Pairs

The following two multistep methods define a predictor-corrector pair

$$L_p\left[y(x_n)\right] = \alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \dots + \alpha_0 y_n - h\left(\beta_{k-1} y'_{n+k-1} + \dots + \beta_0 y'_n\right) \tag{32}$$

$$L_c\left[y(x_n)\right] = \gamma_k y_{n+k} + \gamma_{k-1} y_{n+k-1} + \dots + \gamma_0 y_n - h\left(\delta_k y'_{n+k} + \delta_{k-1} y'_{n+k-1} + \dots \delta_0 y'_n\right) \tag{33}$$

The predictor is used first to give a value for y_{n+k}, which is then used to evaluate y'_{n+k}, which in turn is used by the corrector to compute a new value for y_{n+k}. This process may be iterated; however, usually the degrees of L_p and L_c are equal, and it is assumed that convergence is reached after one application of the corrector. The difference between y_{n+k} of the predictor and y_{n+k} of the corrector is used as an estimate of the local truncation error (ref. 5). In particular,

$$L_p\left[y(x_n)\right] = C_{p,\,d+1} h^{d+1} y^{(d+1)}(x_n) + 0\left(h^{(d+2)}\right) \tag{34}$$

and

$$L_c\left[y(x_n)\right] = C_{c,\,d+1} h^{d+1} y^{(d+1)}(x_n) + 0\left(h^{(d+2)}\right) \tag{35}$$

It follows that

$$y_{c,\,n+k} - y_{p,\,n+k} \simeq \left(C_{c,\,d+1} - C_{p,\,d+1}\right) h^{d+1} y^{(d+1)}(x_n) \tag{36}$$

or

$$\frac{C_{c,d+1}}{C_{c,d+1} - C_{p,d+1}}\left(y_{c,n+k} - y_{p,n+k}\right) \simeq C_{c,d+1} h^{d+1} y^{(d+1)}\left(x_n\right) \tag{37}$$

The advantage of such an error estimate makes it worthwhile to have the degree of the predictor equal to the degree of the corrector.

The characteristic equation of the pair can be derived (ref. 6) by letting $y' = \lambda y$, $\bar{h} = \lambda h$. Then from the predictor

$$y_{n+k} = -\frac{1}{\alpha_k}\left[\left(\alpha_{k-1} - \bar{h}\beta_{k-1}\right) y_{n+k-1} + \ldots + \left(\alpha_0 - \bar{h}\beta_0\right) y_n\right] \tag{38}$$

By substituting this value into the corrector (assuming that $\alpha_k = 1$)

$$\begin{aligned} &\gamma_k y_{n+k} + \gamma_{k-1} y_{n+k-1} + \ldots + \gamma_0 y_n \\ &\quad = \bar{h}\left\{-\delta_k\left[\left(\alpha_{k-1} - \bar{h}\beta_{k-1}\right) y_{n+k-1} + \ldots + \left(\alpha_0 - \bar{h}\beta_0\right) y_n\right]\right. \\ &\qquad \left. + \delta_{k-1} y_{n+k-1} + \ldots + \delta_0 y_n\right\} \end{aligned} \tag{39}$$

and then the characteristic equation is

$$\begin{aligned} &\left(\gamma_k\right) z^k + \left[\gamma_{k-1} + \bar{h}\delta_k\left(\alpha_{k-1} - \bar{h}\beta_{k-1}\right) - \bar{h}\delta_{k-1}\right] z^{k-1} \\ &\qquad + \ldots + \left[\gamma_0 + \bar{h}\delta_k\left(\alpha_0 - \bar{h}\beta_0\right) - \bar{h}\delta_0\right] = 0 \end{aligned} \tag{40}$$

Again it is desirable to rewrite this equation in powers of $\bar{h}$ and compute the boundary of the region of stability by computing the values of $\bar{h}$ for which the characteristic equation has roots of an absolute value equal to 1. The equation in powers of $\bar{h}$ is

$$\left(\delta k \sum_{\ell=0}^{k-1} \beta_\ell z^\ell\right)(\bar{h})^2 + \left[\sum_{\ell=0}^{k-1} \left(\delta_\ell - \delta_k \alpha_\ell\right) z^\ell\right] \bar{h} - \sum_{\ell=0}^{k} \gamma_\ell z^\ell = 0 \tag{41}$$

Let

$$A = \delta_k \sum_{\ell=0}^{k-1} \beta_\ell e^{i\ell\theta}$$

$$B = \sum_{\ell=0}^{k-1} \left(\delta_\ell - \delta_k \alpha_\ell\right) e^{i\ell\theta}$$

$$C = -\sum_{\ell=0}^{k} \gamma_\ell e^{i\ell\theta}$$

Then

$$\bar{h}_1(\theta) = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{42}$$

and

$$\bar{h}_2(\theta) = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{43}$$

By plotting $\bar{h}_1(\theta)$ and $\bar{h}_2(\theta)$ for $0 \leq \theta \leq \pi$ and the conjugates, the region of stability for the predictor-corrector is easily found in exactly the manner described for single formulas.

Example - The Adams-Bashforth $k = 1, d = 1$ predictor and the Adams-Moulton $k = 1, d = 1$ corrector are

$$y_{n+1} - y_n = hy'_n \tag{44}$$

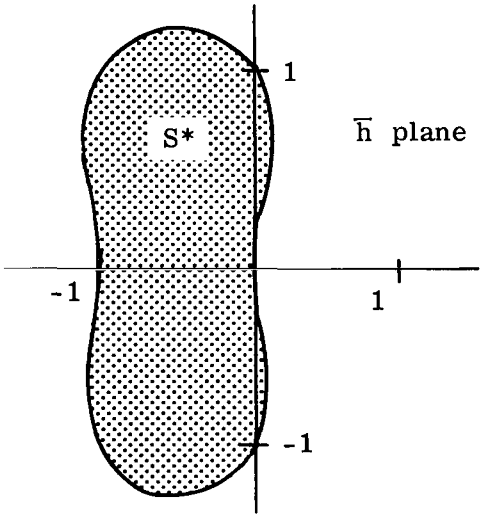
and

$$y_{n+1} - y_n = hy'_{n+1} \tag{45}$$

The joint characteristic equation is

$$(\bar{h})^2 + \bar{h} - e^{i\theta} + 1 = 0 \tag{46}$$

shown in sketch c and



Sketch c

$$\bar{h}_1(\theta) = \frac{-1 + \sqrt{1 + 4\left(e^{i\theta} - 1\right)}}{2} \tag{47}$$

$$\bar{h}_2(\theta) = \frac{-1 - \sqrt{1 + 4\left(e^{i\theta} - 1\right)}}{2} \tag{48}$$

A FORTRAN IV subroutine STBL2 has been written to compute the boundary of the region of stability for predictor-corrector multistep methods. The subroutine is described in appendix B along with examples of the Bashforth-Moulton pairs.

The generalization of this method to finding the region of stability of predictor-corrector pairs where the corrector is iterated m times is apparent. The characteristic equation will be a polynomial of degree $m + 1$ in $\bar{h}$. The details will not be discussed here.

CONCLUDING REMARKS

Algorithms have been developed for calculating the region of stability for single multistep methods and for predictor-corrector pairs. The results are easily generalized to include other versions of the predictor-corrector methods.

It should be noted that, in both cases considered, an explicit formula is given for the boundary of the region of stability. This formula may be useful in developing a technique to generate multistep methods with an increased region of stability.

Manned Spacecraft Center
National Aeronautics and Space Administration
Houston, Texas, January 19, 1967
039-00-00-00-72

APPENDIX A

FORTRAN IV SUBROUTINE STBL1 FOR SINGLE MULTISTEP METHODS

The subroutine STBL1 computes the boundary of the region of stability for a single multistep method of either the predictor or corrector type.

In order to use the subroutine STBL1, the following calling sequence is necessary:

CALL STBL1 (AA, BB, K, N, TITLE)
Where the multistep method is

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = h\left(\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \cdots + \beta_0 f_n\right)$$

and

AA is a double precision array with 25 locations such that $AA(1) = \alpha_k$, $AA(2) = \alpha_{k-1}, \ldots, AA(k + 1) = \alpha_0$.

BB is a double precision array with 25 locations such that $BB(1) = \beta_k$, $BB(2) = \beta_{k-1}, \ldots, BB(k + 1) = \beta_0$.

K is an integer giving the order of the method.

N is an integer giving the number of divisions of the interval $(0, \Pi)$ to be used in computing the boundary of the region of stability; 90 divisions are usually sufficient.

TITLE is an array of 5 locations containing BCD information to be used as a title for the printed and plotted output. TITLE may be passed to the subroutine as 30H (any 30 BCD characters).

STBL1 will compute the boundary of the region of stability and print the results on the line printer if IØ = 6 or on the 4020 output if IØ = 17; it will also plot the results on the 4020 plotter.

Table AI is a listing of the FORTRAN IV subroutine STBL1. Subroutine STBL1 has been used to calculate the regions of stability for the Adams-Bashforth predictors and the Adams-Moulton correctors of degree 1 to 4. Table AII gives the coefficients of the Bashforth methods and the numerical results of STBL1, while figure A1 displays the results graphically. Table AIII gives the coefficients of the Moulton methods and the numerical results of STBL1, while figure A2 displays the results graphically. The coefficients for both of these methods are available from NASA Manned Spacecraft Center, Houston, Texas.

TABLE AI. - LISTING OF FORTRAN IV SUBROUTINE STBL1 FOR

SINGLE MULTISTEP METHODS

```
      DIMENSION AA(25), BB(25)
      DOUBLE PRECISION AA, BB
      AA(1) = 1.0D0
      AA(2) = -1.0D0
      DO 1025 J = 3,25
 1025 AA(J) = 0.0D0
      BB(1) = 0.0D0
      BB(2) = 1.0D0
      KSTEP = 1
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS BASHFORTH,  K=1,  D=1     )
      BB(1) = 0.0D0
       BB(2) =  0.15000000000000000D1
       BB(3) = -0.50000000000000000D0
      KSTEP = 2
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS BASHFORTH,  K=2,  D=2     )
      BB(1) = 0.0D0
       BB(2) =  0.19166666666666667D1
       BB(3) = -0.13333333333333333D1
       BB(4) =  0.41666666666666667D0
      KSTEP = 3
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS BASHFORTH,  K=3,  D=3     )
      BB(1) = 0.0D0
       BB(2) =  0.22916666666666667D1
       BB(3) = -0.24583333333333333D1
       BB(4) =  0.15416666666666667D1
       BB(5) = -0.37500000000000000D0
      KSTEP = 4
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS BASHFORTH,  K=4,  D=4     )
      BB(1) = 1.0D0
      BB(2) = 0.0D0
      KSTEP = 1
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS MOULTON  K=1,  D=1        )
       BB(1) = 0.5D0
       BB(2) = 0.5D0
      KSTEP = 1
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS MOULTON  K=1,  D=2        )
       BB(1) = 0.41666666666666667D-00
       BB(2) = 0.66666666666666667D-00
       BB(3) =-0.83333333333333333D-01
      KSTEP = 2
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS MOULTON  K=2,  D=3        )
       BB(1) =  0.375D0
       BB(2) =  0.79166666666666667D-00
       BB(3) = -0.20833333333333333D-00
       BB(4) =  0.41666666666666667D-01
      KSTEP = 3
      CALL STBL1(AA,BB,KSTEP,100,
     1            30HADAMS MOULTON  K=3,  D=4        )
      END
```

TABLE AI. - LISTING OF FORTRAN IV SUBROUTINE STBL1 FOR SINGLE MULTISTEP METHODS - Concluded

```
      SUBROUTINE STBL1(AA, BB, K, N, TITLE)
      DIMENSION AA(25), BB(25), H1(500), H2(500), TITLE(5)
      DOUBLE PRECISION AA, BB, X1, X2, Y1, Y2, DSIN, DCOS, DEL, THETA
      DOUBLE PRECISION DL, D1, D2, DD
      DIMENSION BCDX(12), BCDY(12)
      DIMENSION TH(500)
      DATA BCDX/12*6H      /
      DATA BCDY/12*6H      /
      IO=6
      IO=17
      K1=K+1
      IC=0
      NC=0
      DEL=3.141592653589793D0/N
      THETA=0.0D0
 1020 IF (NC .GT. N) GO TO 1030
      THETA = NC*DEL
      X1=0.0D0
      X2=0.0D0
      Y1=0.0D0
      Y2=0.0D0
      DO 1010 I=1,K1
      L=K-I+1
      DL=L
      X1=X1+AA(I)*DCOS(DL*THETA)
      X2=X2+AA(I)*DSIN(DL*THETA)
      Y1=Y1+BB(I)*DCOS(DL*THETA)
      Y2=Y2+BB(I)*DSIN(DL*THETA)
 1010 CONTINUE
      DD=Y1**2+Y2**2
      IF(DD .LT. 1.0D-5) GO TO 1040
      D1=(X1*Y1+X2*Y2)/DD
      D2=(X2*Y1-X1*Y2)/DD
      IC=IC+1
      H1(IC)=D1
      H2(IC)=D2
      TH(IC) = THETA
 1040 NC=NC+1
      GO TO 1020
 1030 CONTINUE
 6001 FORMAT(1H1,40X,4H--- ,5A6,4H ---,///)
      WRITE(IO,6001) (TITLE(J),J=1,5)
      DO 1060 J=1,K1
      L=K-J+1
 1060 WRITE(IO,6002) L,AA(J),L,BB(J)
 6002 FORMAT(1H ,15X,6HALPHA(,I2,4H) = ,D21.16,20X,
     1                6H BETA(,I2,4H) = ,D21.16)
 6005 FORMAT(///)
      WRITE(IO,6005)
 6003 FORMAT(1H ,13X,E10.4,5X,1H(,E10.4,3H , F10.4,1H),10X,
     1                F10.4,5X,1H(,E10.4,3H , F10.4,1H))
 6006 FORMAT(1H ,15X,5HTHETA,18X,4HHBAR,23X,5HTHETA,18X,4HHBAR/)
      WRITE(IO,6006)
      IC2=(IC+1)/2
      DO 1070 I=1,IC2
      J=I+IC/2
      WRITE(IO,6003) (TH(I),H1(I),H2(I),TH(J),H1(J),H2(J))
 1070 CONTINUE
      DO 1080 I=1,5
 1080 BCDX(I) = TITLE(I)
      CALL QUIKML(-1,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H1,H2)
      DO 1090 I=1,IC
 1090 H2(I) =-H2(I)
      CALL QUIKML(0,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H1,H2)
      RETURN
      END
```

TABLE AII. - LINE PRINTOUTS OF ADAMS-BASHFORTH PREDICTORS

(a) k = 1, d = 1

--- ADAMS BASHFORTH, K=1, D=1 ---

ALPHA(1) = .1000000000000000+01		BETA(1) = .0000000000000000
ALPHA(0) = -.1000000000000000+01		BETA(0) = .1000000000000000+01

THETA	HBAR		THETA	HBAR	
.0000	(.0000 ,	-.0000)	.1571+01	(-.1000+01 ,	.1000+01)
.3142-01	(-.4934-03 ,	.3141-01)	.1602+01	(-.1031+01 ,	.9995-00)
.6283-01	(-.1973-02 ,	.6279-01)	.1634+01	(-.1063+01 ,	.9980-00)
.9425-01	(-.4438-02 ,	.9411-01)	.1665+01	(-.1094+01 ,	.9956-00)
.1257-00	(-.7885-02 ,	.1253-00)	.1696+01	(-.1125+01 ,	.9921-00)
.1571-00	(-.1231-01 ,	.1564-00)	.1728+01	(-.1156+01 ,	.9877-00)
.1885-00	(-.1771-01 ,	.1874-00)	.1759+01	(-.1187+01 ,	.9823-00)
.2199-00	(-.2408-01 ,	.2181-00)	.1791+01	(-.1218+01 ,	.9759-00)
.2513-00	(-.3142-01 ,	.2487-00)	.1822+01	(-.1249+01 ,	.9686-00)
.2827-00	(-.3971-01 ,	.2790-00)	.1854+01	(-.1279+01 ,	.9603-00)
.3142-00	(-.4894-01 ,	.3090-00)	.1885+01	(-.1309+01 ,	.9511-00)
.3456-00	(-.5912-01 ,	.3387-00)	.1916+01	(-.1339+01 ,	.9409-00)
.3770-00	(-.7022-01 ,	.3681-00)	.1948+01	(-.1368+01 ,	.9298-00)
.4084-00	(-.8225-01 ,	.3971-00)	.1979+01	(-.1397+01 ,	.9178-00)
.4398-00	(-.9517-01 ,	.4258-00)	.2011+01	(-.1426+01 ,	.9048-00)
.4712-00	(-.1090+00 ,	.4540-00)	.2042+01	(-.1454+01 ,	.8910-00)
.5027-00	(-.1237+00 ,	.4818-00)	.2073+01	(-.1482+01 ,	.8763-00)
.5341-00	(-.1393-00 ,	.5090-00)	.2105+01	(-.1509+01 ,	.8607-00)
.5655-00	(-.1557-00 ,	.5358-00)	.2136+01	(-.1536+01 ,	.8443-00)
.5969-00	(-.1729-00 ,	.5621-00)	.2168+01	(-.1562+01 ,	.8271-00)
.6283-00	(-.1910-00 ,	.5878-00)	.2199+01	(-.1588+01 ,	.8090-00)
.6597-00	(-.2098-00 ,	.6129-00)	.2231+01	(-.1613+01 ,	.7902-00)
.6912-00	(-.2295-00 ,	.6374-00)	.2262+01	(-.1637+01 ,	.7705-00)
.7226-00	(-.2499-00 ,	.6613-00)	.2293+01	(-.1661+01 ,	.7501-00)
.7540-00	(-.2710-00 ,	.6845-00)	.2325+01	(-.1685+01 ,	.7290-00)
.7854-00	(-.2929-00 ,	.7071-00)	.2356+01	(-.1707+01 ,	.7071-00)
.8168-00	(-.3155-00 ,	.7290-00)	.2388+01	(-.1729+01 ,	.6845-00)
.8482-00	(-.3387-00 ,	.7501-00)	.2419+01	(-.1750+01 ,	.6613-00)
.8796-00	(-.3626-00 ,	.7705-00)	.2450+01	(-.1771+01 ,	.6374-00)
.9111-00	(-.3871-00 ,	.7902-00)	.2482+01	(-.1790+01 ,	.6129-00)
.9425-00	(-.4122-00 ,	.8090-00)	.2513+01	(-.1809+01 ,	.5878-00)
.9739-00	(-.4379-00 ,	.8271-00)	.2545+01	(-.1827+01 ,	.5621-00)
.1005+01	(-.4642-00 ,	.8443-00)	.2576+01	(-.1844+01 ,	.5358-00)
.1037+01	(-.4910-00 ,	.8607-00)	.2608+01	(-.1861+01 ,	.5090-00)
.1068+01	(-.5182-00 ,	.8763-00)	.2639+01	(-.1876+01 ,	.4818-00)
.1100+01	(-.5460-00 ,	.8910-00)	.2670+01	(-.1891+01 ,	.4540-00)
.1131+01	(-.5742-00 ,	.9048-00)	.2702+01	(-.1905+01 ,	.4258-00)
.1162+01	(-.6029-00 ,	.9178-00)	.2733+01	(-.1918+01 ,	.3971-00)
.1194+01	(-.6319-00 ,	.9298-00)	.2765+01	(-.1930+01 ,	.3681-00)
.1225+01	(-.6613-00 ,	.9409-00)	.2796+01	(-.1941+01 ,	.3387-00)
.1257+01	(-.6910-00 ,	.9511-00)	.2827+01	(-.1951+01 ,	.3090-00)
.1288+01	(-.7210-00 ,	.9603-00)	.2859+01	(-.1960+01 ,	.2790-00)
.1319+01	(-.7513-00 ,	.9686-00)	.2890+01	(-.1969+01 ,	.2487-00)
.1351+01	(-.7819-00 ,	.9759-00)	.2922+01	(-.1976+01 ,	.2181-00)
.1382+01	(-.8126-00 ,	.9823-00)	.2953+01	(-.1982+01 ,	.1874-00)
.1414+01	(-.8436-00 ,	.9877-00)	.2985+01	(-.1988+01 ,	.1564-00)
.1445+01	(-.8747-00 ,	.9921-00)	.3016+01	(-.1992+01 ,	.1253-00)
.1477+01	(-.9059-00 ,	.9956-00)	.3047+01	(-.1996+01 ,	.9411-01)
.1508+01	(-.9372-00 ,	.9980-00)	.3079+01	(-.1998+01 ,	.6279-01)
.1539+01	(-.9686-00 ,	.9995-00)	.3110+01	(-.2000+01 ,	.3141-01)
.1571+01	(-.1000+01 ,	.1000+01)	.3142+01	(-.2000+01 ,	.9992-15)

TABLE AII. - LINE PRINTOUTS OF ADAMS-BASHFORTH PREDICTORS - Continued

(b) k = 2, d = 2

--- ADAMS BASHFORTH, K=2, D=2 ---

```
ALPHA( 2) =  .1000000000000000+01          BETA( 2) =  .0000000000000000
ALPHA( 1) = -.1000000000000000+01          BETA( 1) =  .1500000000000000+01
ALPHA( 0) =  .0000000000000000             BETA( 0) = -.5000000000000000+00
```

THETA	HBAR	THETA	HBAR
.0000	(.0000 , -.0000)	.1571+01	(-.4000-00 , .8000-00)
.3142-01	(-.2433-06 , .3140-01)	.1602+01	(-.4177-00 , .7971-00)
.6283-01	(-.3882-05 , .6273-01)	.1634+01	(-.4354-00 , .7936-00)
.9425-01	(-.1957-04 , .9390-01)	.1665+01	(-.4532-00 , .7894-00)
.1257-00	(-.6145-04 , .1248+00)	.1696+01	(-.4711-00 , .7844-00)
.1571-00	(-.1488-03 , .1555-00)	.1728+01	(-.4890-00 , .7789-00)
.1885-00	(-.3056-03 , .1858-00)	.1759+01	(-.5070-00 , .7726-00)
.2199-00	(-.5598-03 , .2156-00)	.1791+01	(-.5249-00 , .7657-00)
.2513-00	(-.9426-03 , .2450-00)	.1822+01	(-.5427-00 , .7581-00)
.2827-00	(-.1488-02 , .2738-00)	.1854+01	(-.5605-00 , .7499-00)
.3142-00	(-.2232-02 , .3020-00)	.1885+01	(-.5782-00 , .7410-00)
.3456-00	(-.3210-02 , .3295-00)	.1916+01	(-.5958-00 , .7315-00)
.3770-00	(-.4461-02 , .3564-00)	.1948+01	(-.6133-00 , .7214-00)
.4084-00	(-.6021-02 , .3826-00)	.1979+01	(-.6306-00 , .7107-00)
.4398-00	(-.7926-02 , .4080-00)	.2011+01	(-.6477-00 , .6993-00)
.4712-00	(-.1021-01 , .4327-00)	.2042+01	(-.6646-00 , .6874-00)
.5027-00	(-.1291-01 , .4566-00)	.2073+01	(-.6813-00 , .6748-00)
.5341-00	(-.1604-01 , .4797-00)	.2105+01	(-.6978-00 , .6617-00)
.5655-00	(-.1965-01 , .5020-00)	.2136+01	(-.7140-00 , .6481-00)
.5969-00	(-.2374-01 , .5235-00)	.2168+01	(-.7299-00 , .6339-00)
.6283-00	(-.2835-01 , .5442-00)	.2199+01	(-.7455-00 , .6191-00)
.6597-00	(-.3349-01 , .5640-00)	.2231+01	(-.7608-00 , .6038-00)
.6912-00	(-.3918-01 , .5830-00)	.2262+01	(-.7758-00 , .5880-00)
.7226-00	(-.4542-01 , .6012-00)	.2293+01	(-.7904-00 , .5717-00)
.7540-00	(-.5223-01 , .6186-00)	.2325+01	(-.8046-00 , .5549-00)
.7854-00	(-.5960-01 , .6352-00)	.2356+01	(-.8184-00 , .5376-00)
.8168-00	(-.6755-01 , .6509-00)	.2388+01	(-.8319-00 , .5199-00)
.8482-00	(-.7607-01 , .6659-00)	.2419+01	(-.8449-00 , .5017-00)
.8796-00	(-.8515-01 , .6800-00)	.2450+01	(-.8575-00 , .4831-00)
.9111-00	(-.9480-01 , .6934-00)	.2482+01	(-.8696-00 , .4640-00)
.9425-00	(-.1050+00 , .7060-00)	.2513+01	(-.8812-00 , .4446-00)
.9739-00	(-.1157+00 , .7178-00)	.2545+01	(-.8924-00 , .4248-00)
.1005+01	(-.1270-00 , .7288-00)	.2576+01	(-.9031-00 , .4046-00)
.1037+01	(-.1388-00 , .7391-00)	.2608+01	(-.9133-00 , .3841-00)
.1068+01	(-.1511-00 , .7485-00)	.2639+01	(-.9229-00 , .3633-00)
.1100+01	(-.1639-00 , .7573-00)	.2670+01	(-.9321-00 , .3421-00)
.1131+01	(-.1771-00 , .7653-00)	.2702+01	(-.9407-00 , .3206-00)
.1162+01	(-.1908-00 , .7725-00)	.2733+01	(-.9487-00 , .2989-00)
.1194+01	(-.2050-00 , .7790-00)	.2765+01	(-.9562-00 , .2769-00)
.1225+01	(-.2195-00 , .7847-00)	.2796+01	(-.9631-00 , .2547-00)
.1257+01	(-.2345-00 , .7897-00)	.2827+01	(-.9694-00 , .2322-00)
.1288+01	(-.2497-00 , .7940-00)	.2859+01	(-.9752-00 , .2096-00)
.1319+01	(-.2654-00 , .7975-00)	.2890+01	(-.9804-00 , .1868-00)
.1351+01	(-.2813-00 , .8003-00)	.2922+01	(-.9850-00 , .1638-00)
.1382+01	(-.2976-00 , .8024-00)	.2953+01	(-.9889-00 , .1406-00)
.1414+01	(-.3141-00 , .8038-00)	.2985+01	(-.9923-00 , .1174+00)
.1445+01	(-.3309-00 , .8044-00)	.3016+01	(-.9951-00 , .9403-01)
.1477+01	(-.3479-00 , .8044-00)	.3047+01	(-.9972-00 , .7059-01)
.1508+01	(-.3651-00 , .8036-00)	.3079+01	(-.9988-00 , .4710-01)
.1539+01	(-.3825-00 , .8022-00)	.3110+01	(-.9997-00 , .2356-01)
.1571+01	(-.4000-00 , .8000-00)	.3142+01	(-.1000+01 , .7494-15)

TABLE AII. - LINE PRINTOUTS OF ADAMS-BASHFORTH PREDICTORS - Continued

(c) k = 3, d = 3

--- ADAMS BASHFORTH, K=3, D=3 ---

```
ALPHA( 3) =  .1000000000000000+01          BETA( 3) =  .0000000000000000
ALPHA( 2) = -.1000000000000000+01          BETA( 2) =  .1916666666666666+01
ALPHA( 1) =  .0000000000000000             BETA( 1) = -.1333333333333332+01
ALPHA( 0) =  .0000000000000000             BETA( 0) =  .4166666666666666+00
```

THETA	HBAR	THETA	HBAR
.0000	(.0000 , -.0000)	.1571+01	(-.4138-01 , .7034-00)
.3142-01	(.3651-06 , .3142-01)	.1602+01	(-.5513-01 , .6965-00)
.6283-01	(.5830-05 , .6283-01)	.1634+01	(-.6916-01 , .6891-00)
.9425-01	(.2942-04 , .9424-01)	.1665+01	(-.8343-01 , .6815-00)
.1257-00	(.9255-04 , .1257-00)	.1696+01	(-.9790-01 , .6736-00)
.1571-00	(.2246-03 , .1570-00)	.1728+01	(-.1125+00 , .6653-00)
.1885-00	(.4623-03 , .1884-00)	.1759+01	(-.1273-00 , .6568-00)
.2199-00	(.8486-03 , .2197-00)	.1791+01	(-.1421-00 , .6479-00)
.2513-00	(.1432-02 , .2509-00)	.1822+01	(-.1570-00 , .6387-00)
.2827-00	(.2264-02 , .2820-00)	.1854+01	(-.1719-00 , .6292-00)
.3142-00	(.3400-02 , .3129-00)	.1885+01	(-.1867-00 , .6194-00)
.3456-00	(.4892-02 , .3436-00)	.1916+01	(-.2016-00 , .6092-00)
.3770-00	(.6792-02 , .3739-00)	.1948+01	(-.2163-00 , .5988-00)
.4084-00	(.9147-02 , .4039-00)	.1979+01	(-.2310-00 , .5879-00)
.4398-00	(.1199-01 , .4333-00)	.2011+01	(-.2455-00 , .5768-00)
.4712-00	(.1536-01 , .4621-00)	.2042+01	(-.2599-00 , .5654-00)
.5027-00	(.1925-01 , .4901-00)	.2073+01	(-.2741-00 , .5536-00)
.5341-00	(.2367-01 , .5173-00)	.2105+01	(-.2881-00 , .5415-00)
.5655-00	(.2859-01 , .5434-00)	.2136+01	(-.3019-00 , .5290-00)
.5969-00	(.3397-01 , .5684-00)	.2168+01	(-.3155-00 , .5163-00)
.6283-00	(.3974-01 , .5921-00)	.2199+01	(-.3288-00 , .5032-00)
.6597-00	(.4581-01 , .6144-00)	.2231+01	(-.3419-00 , .4898-00)
.6912-00	(.5208-01 , .6352-00)	.2262+01	(-.3546-00 , .4761-00)
.7226-00	(.5844-01 , .6544-00)	.2293+01	(-.3671-00 , .4621-00)
.7540-00	(.6474-01 , .6720-00)	.2325+01	(-.3792-00 , .4478-00)
.7854-00	(.7085-01 , .6878-00)	.2356+01	(-.3910-00 , .4332-00)
.8168-00	(.7662-01 , .7020-00)	.2388+01	(-.4025-00 , .4183-00)
.8482-00	(.8192-01 , .7144-00)	.2419+01	(-.4136-00 , .4032-00)
.8796-00	(.8661-01 , .7252-00)	.2450+01	(-.4243-00 , .3877-00)
.9111-00	(.9058-01 , .7343-00)	.2482+01	(-.4346-00 , .3720-00)
.9425-00	(.9371-01 , .7420-00)	.2513+01	(-.4446-00 , .3561-00)
.9739-00	(.9593-01 , .7481-00)	.2545+01	(-.4541-00 , .3399-00)
.1005+01	(.9715-01 , .7529-00)	.2576+01	(-.4632-00 , .3234-00)
.1037+01	(.9734-01 , .7564-00)	.2608+01	(-.4718-00 , .3068-00)
.1068+01	(.9646-01 , .7588-00)	.2639+01	(-.4800-00 , .2899-00)
.1100+01	(.9450-01 , .7601-00)	.2670+01	(-.4878-00 , .2728-00)
.1131+01	(.9145-01 , .7605-00)	.2702+01	(-.4951-00 , .2555-00)
.1162+01	(.8735-01 , .7600-00)	.2733+01	(-.5019-00 , .2380-00)
.1194+01	(.8222-01 , .7587-00)	.2765+01	(-.5083-00 , .2204-00)
.1225+01	(.7609-01 , .7567-00)	.2796+01	(-.5142-00 , .2026-00)
.1257+01	(.6901-01 , .7541-00)	.2827+01	(-.5195-00 , .1847-00)
.1288+01	(.6104-01 , .7509-00)	.2859+01	(-.5244-00 , .1666-00)
.1319+01	(.5224-01 , .7472-00)	.2890+01	(-.5288-00 , .1484-00)
.1351+01	(.4265-01 , .7431-00)	.2922+01	(-.5327-00 , .1301-00)
.1382+01	(.3234-01 , .7385-00)	.2953+01	(-.5361-00 , .1117+00)
.1414+01	(.2137-01 , .7336-00)	.2985+01	(-.5389-00 , .9318-01)
.1445+01	(.9794-02 , .7282-00)	.3016+01	(-.5413-00 , .7463-01)
.1477+01	(-.2322-02 , .7225-00)	.3047+01	(-.5431-00 , .5602-01)
.1508+01	(-.1493-01 , .7165-00)	.3079+01	(-.5444-00 , .3737-01)
.1539+01	(-.2796-01 , .7101-00)	.3110+01	(-.5452-00 , .1869-01)
.1571+01	(-.4138-01 , .7034-00)	.3142+01	(-.5455-00 , .6097-15)

TABLE AII. - LINE PRINTOUTS OF ADAMS-BASHFORTH PREDICTORS - Concluded

(d) k = 4, d = 4

--- ADAMS BASHFORTH, K=4, D=4 ---

ALPHA(4) =	.1000000000000000+01	BETA(4) =	.0000000000000000
ALPHA(3) =	-.1000000000000000+01	BETA(3) =	.2291666666666666+01
ALPHA(2) =	.0000000000000000	BETA(2) =	-.2458333333333332+01
ALPHA(1) =	.0000000000000000	BETA(1) =	.1541666666666666+01
ALPHA(0) =	.0000000000000000	BETA(0) =	-.3750000000000000+00

THETA	HBAR	
.0000	(.0000 ,	-.0000)
.3142-01	(.5205-09 ,	.3142-01)
.6283-01	(.3326-07 ,	.6283-01)
.9425-01	(.3779-06 ,	.9425-01)
.1257-00	(.2116-05 ,	.1257-00)
.1571-00	(.8038-05 ,	.1571-00)
.1885-00	(.2388-04 ,	.1886-00)
.2199-00	(.5985-04 ,	.2201-00)
.2513-00	(.1325-03 ,	.2516-00)
.2827-00	(.2666-03 ,	.2833-00)
.3142-00	(.4978-03 ,	.3151-00)
.3456-00	(.8747-03 ,	.3470-00)
.3770-00	(.1462-02 ,	.3792-00)
.4084-00	(.2341-02 ,	.4116-00)
.4398-00	(.3617-02 ,	.4442-00)
.4712-00	(.5418-02 ,	.4771-00)
.5027-00	(.7898-02 ,	.5104-00)
.5341-00	(.1124-01 ,	.5439-00)
.5655-00	(.1565-01 ,	.5778-00)
.5969-00	(.2137-01 ,	.6118-00)
.6283-00	(.2867-01 ,	.6459-00)
.6597-00	(.3782-01 ,	.6799-00)
.6912-00	(.4910-01 ,	.7136-00)
.7226-00	(.6280-01 ,	.7465-00)
.7540-00	(.7914-01 ,	.7783-00)
.7854-00	(.9828-01 ,	.8084-00)
.8168-00	(.1203+00 ,	.8362-00)
.8482-00	(.1450-00 ,	.8609-00)
.8796-00	(.1722-00 ,	.8817-00)
.9111-00	(.2013-00 ,	.8981-00)
.9425-00	(.2317-00 ,	.9094-00)
.9739-00	(.2624-00 ,	.9151-00)
.1005+01	(.2924-00 ,	.9152-00)
.1037+01	(.3207-00 ,	.9096-00)
.1068+01	(.3462-00 ,	.8989-00)
.1100+01	(.3682-00 ,	.8836-00)
.1131+01	(.3859-00 ,	.8647-00)
.1162+01	(.3991-00 ,	.8429-00)
.1194+01	(.4076-00 ,	.8194-00)
.1225+01	(.4115-00 ,	.7948-00)
.1257+01	(.4112-00 ,	.7700-00)
.1288+01	(.4071-00 ,	.7456-00)
.1319+01	(.3996-00 ,	.7220-00)
.1351+01	(.3893-00 ,	.6994-00)
.1382+01	(.3766-00 ,	.6782-00)
.1414+01	(.3619-00 ,	.6582-00)
.1445+01	(.3457-00 ,	.6397-00)
.1477+01	(.3284-00 ,	.6225-00)
.1508+01	(.3101-00 ,	.6065-00)
.1539+01	(.2913-00 ,	.5917-00)
.1571+01	(.2720-00 ,	.5779-00)
.1571+01	(.2720-00 ,	.5779-00)
.1602+01	(.2524-00 ,	.5650-00)
.1634+01	(.2327-00 ,	.5529-00)
.1665+01	(.2130-00 ,	.5415-00)
.1696+01	(.1933-00 ,	.5306-00)
.1728+01	(.1738-00 ,	.5202-00)
.1759+01	(.1545-00 ,	.5102-00)
.1791+01	(.1354-00 ,	.5005-00)
.1822+01	(.1166+00 ,	.4911-00)
.1854+01	(.9814-01 ,	.4818-00)
.1885+01	(.7997-01 ,	.4726-00)
.1916+01	(.6214-01 ,	.4635-00)
.1948+01	(.4467-01 ,	.4544-00)
.1979+01	(.2756-01 ,	.4453-00)
.2011+01	(.1083-01 ,	.4361-00)
.2042+01	(-.5516-02 ,	.4268-00)
.2073+01	(-.2147-01 ,	.4175-00)
.2105+01	(-.3703-01 ,	.4080-00)
.2136+01	(-.5218-01 ,	.3983-00)
.2168+01	(-.6693-01 ,	.3885-00)
.2199+01	(-.8126-01 ,	.3786-00)
.2231+01	(-.9518-01 ,	.3684-00)
.2262+01	(-.1087+00 ,	.3581-00)
.2293+01	(-.1217+00 ,	.3475-00)
.2325+01	(-.1344-00 ,	.3368-00)
.2356+01	(-.1466-00 ,	.3259-00)
.2388+01	(-.1584-00 ,	.3147-00)
.2419+01	(-.1697-00 ,	.3034-00)
.2450+01	(-.1806-00 ,	.2918-00)
.2482+01	(-.1910-00 ,	.2801-00)
.2513+01	(-.2010-00 ,	.2682-00)
.2545+01	(-.2105-00 ,	.2560-00)
.2576+01	(-.2195-00 ,	.2437-00)
.2608+01	(-.2281-00 ,	.2312-00)
.2639+01	(-.2363-00 ,	.2186-00)
.2670+01	(-.2439-00 ,	.2058-00)
.2702+01	(-.2511-00 ,	.1928-00)
.2733+01	(-.2578-00 ,	.1797-00)
.2765+01	(-.2640-00 ,	.1664-00)
.2796+01	(-.2697-00 ,	.1530-00)
.2827+01	(-.2749-00 ,	.1395-00)
.2859+01	(-.2797-00 ,	.1258-00)
.2890+01	(-.2839-00 ,	.1121+00)
.2922+01	(-.2877-00 ,	.9831-01)
.2953+01	(-.2910-00 ,	.8441-01)
.2985+01	(-.2937-00 ,	.7045-01)
.3016+01	(-.2960-00 ,	.5643-01)
.3047+01	(-.2977-00 ,	.4236-01)
.3079+01	(-.2990-00 ,	.2826-01)
.3110+01	(-.2997-00 ,	.1414-01)
.3142+01	(-.3000-00 ,	.4606-15)

TABLE AIII. - LINE PRINTOUTS OF ADAMS-MOULTON CORRECTORS

(a) k = 1, d = 1

--- ADAMS MOULTON K=1, D=1 ---

```
ALPHA( 1) =  .1000000000000000+01          BETA( 1) =  .1000000000000000+01
ALPHA( 0) = -.1000000000000000+01          BETA( 0) =  .0000000000000000
```

THETA	HBAR	THETA	HBAR
.0000	(.0000 , -.0000)	.1571+01	(.1000+01 , .1000+01)
.3142-01	(.4934-03 , .3141-01)	.1602+01	(.1031+01 , .9995-00)
.6283-01	(.1973-02 , .6279-01)	.1634+01	(.1063+01 , .9980-00)
.9425-01	(.4438-02 , .9411-01)	.1665+01	(.1094+01 , .9956-00)
.1257-00	(.7885-02 , .1253-00)	.1696+01	(.1125+01 , .9921-00)
.1571-00	(.1231-01 , .1564-00)	.1728+01	(.1156+01 , .9877-00)
.1885-00	(.1771-01 , .1874-00)	.1759+01	(.1187+01 , .9823-00)
.2199-00	(.2408-01 , .2181-00)	.1791+01	(.1218+01 , .9759-00)
.2513-00	(.3142-01 , .2487-00)	.1822+01	(.1249+01 , .9686-00)
.2827-00	(.3971-01 , .2790-00)	.1854+01	(.1279+01 , .9603-00)
.3142-00	(.4894-01 , .3090-00)	.1885+01	(.1309+01 , .9511-00)
.3456-00	(.5912-01 , .3387-00)	.1916+01	(.1339+01 , .9409-00)
.3770-00	(.7022-01 , .3681-00)	.1948+01	(.1368+01 , .9298-00)
.4084-00	(.8225-01 , .3971-00)	.1979+01	(.1397+01 , .9178-00)
.4398-00	(.9517-01 , .4258-00)	.2011+01	(.1426+01 , .9048-00)
.4712-00	(.1090+00 , .4540-00)	.2042+01	(.1454+01 , .8910-00)
.5027-00	(.1237+00 , .4818-00)	.2073+01	(.1482+01 , .8763-00)
.5341-00	(.1393-00 , .5090-00)	.2105+01	(.1509+01 , .8607-00)
.5655-00	(.1557-00 , .5358-00)	.2136+01	(.1536+01 , .8443-00)
.5969-00	(.1729-00 , .5621-00)	.2168+01	(.1562+01 , .8271-00)
.6283-00	(.1910-00 , .5878-00)	.2199+01	(.1588+01 , .8090-00)
.6597-00	(.2098-00 , .6129-00)	.2231+01	(.1613+01 , .7902-00)
.6912-00	(.2295-00 , .6374-00)	.2262+01	(.1637+01 , .7705-00)
.7226-00	(.2499-00 , .6613-00)	.2293+01	(.1661+01 , .7501-00)
.7540-00	(.2710-00 , .6845-00)	.2325+01	(.1685+01 , .7290-00)
.7854-00	(.2929-00 , .7071-00)	.2356+01	(.1707+01 , .7071-00)
.8168-00	(.3155-00 , .7290-00)	.2388+01	(.1729+01 , .6845-00)
.8482-00	(.3387-00 , .7501-00)	.2419+01	(.1750+01 , .6613-00)
.8796-00	(.3626-00 , .7705-00)	.2450+01	(.1771+01 , .6374-00)
.9111-00	(.3871-00 , .7902-00)	.2482+01	(.1790+01 , .6129-00)
.9425-00	(.4122-00 , .8090-00)	.2513+01	(.1809+01 , .5878-00)
.9739-00	(.4379-00 , .8271-00)	.2545+01	(.1827+01 , .5621-00)
.1005+01	(.4642-00 , .8443-00)	.2576+01	(.1844+01 , .5358-00)
.1037+01	(.4910-00 , .8607-00)	.2608+01	(.1861+01 , .5090-00)
.1068+01	(.5182-00 , .8763-00)	.2639+01	(.1876+01 , .4818-00)
.1100+01	(.5460-00 , .8910-00)	.2670+01	(.1891+01 , .4540-00)
.1131+01	(.5742-00 , .9048-00)	.2702+01	(.1905+01 , .4258-00)
.1162+01	(.6029-00 , .9178-00)	.2733+01	(.1918+01 , .3971-00)
.1194+01	(.6319-00 , .9298-00)	.2765+01	(.1930+01 , .3681-00)
.1225+01	(.6613-00 , .9409-00)	.2796+01	(.1941+01 , .3387-00)
.1257+01	(.6910-00 , .9511-00)	.2827+01	(.1951+01 , .3090-00)
.1288+01	(.7210-00 , .9603-00)	.2859+01	(.1960+01 , .2790-00)
.1319+01	(.7513-00 , .9686-00)	.2890+01	(.1969+01 , .2487-00)
.1351+01	(.7819-00 , .9759-00)	.2922+01	(.1976+01 , .2181-00)
.1382+01	(.8126-00 , .9823-00)	.2953+01	(.1982+01 , .1874-00)
.1414+01	(.8436-00 , .9877-00)	.2985+01	(.1988+01 , .1564-00)
.1445+01	(.8747-00 , .9921-00)	.3016+01	(.1992+01 , .1253-00)
.1477+01	(.9059-00 , .9956-00)	.3047+01	(.1996+01 , .9411-01)
.1508+01	(.9372-00 , .9980-00)	.3079+01	(.1998+01 , .6279-01)
.1539+01	(.9686-00 , .9995-00)	.3110+01	(.2000+01 , .3141-01)
.1571+01	(.1000+01 , .1000+01)	.3142+01	(.2000+01 , .9992-15)

TABLE AIII. - LINE PRINTOUTS OF ADAMS-MOULTON CORRECTORS - Continued

(b) k = 1, d = 2

--- ADAMS MOULTON K=1, D=2 ---

ALPHA(1) = .1000000000000000+01
ALPHA(0) = -.1000000000000000+01

BETA(1) = .5000000000000000+00
BETA(0) = .5000000000000000+00

THETA	HBAR	
.0000	(.0000 ,	-.0000)
.3142-01	(-.6984-16 ,	.3142-01)
.6283-01	(-.1189-15 ,	.6285-01)
.9425-01	(-.2260-16 ,	.9432-01)
.1257-00	(-.7924-16 ,	.1258-00)
.1571-00	(.2618-16 ,	.1574-00)
.1885-00	(.2450-16 ,	.1891-00)
.2199-00	(.8779-17 ,	.2208-00)
.2513-00	(-.7050-16 ,	.2527-00)
.2827-00	(-.6371-16 ,	.2846-00)
.3142-00	(-.8891-16 ,	.3168-00)
.3456-00	(-.8223-16 ,	.3491-00)
.3770-00	(-.1223-15 ,	.3815-00)
.4084-00	(-.1013-15 ,	.4142-00)
.4398-00	(-.2914-16 ,	.4471-00)
.4712-00	(-.8807-16 ,	.4802-00)
.5027-00	(.7396-17 ,	.5135-00)
.5341-00	(-.1492-16 ,	.5471-00)
.5655-00	(-.6020-16 ,	.5811-00)
.5969-00	(-.1671-15 ,	.6153-00)
.6283-00	(-.1381-15 ,	.6498-00)
.6597-00	(-.1240-15 ,	.6848-00)
.6912-00	(-.1254-15 ,	.7200-00)
.7226-00	(-.6344-16 ,	.7557-00)
.7540-00	(-.1605-15 ,	.7919-00)
.7854-00	(-.1301-15 ,	.8284-00)
.8168-00	(-.1977-15 ,	.8655-00)
.8482-00	(-.1671-15 ,	.9030-00)
.8796-00	(-.6780-16 ,	.9411-00)
.9111-00	(-.1377-15 ,	.9798-00)
.9425-00	(-.2447-15 ,	.1019+01)
.9739-00	(-.1777-15 ,	.1059+01)
.1005+01	(-.1807-15 ,	.1100+01)
.1037+01	(-.7357-16 ,	.1141+01)
.1068+01	(-.1873-15 ,	.1183+01)
.1100+01	(-.3818-16 ,	.1226+01)
.1131+01	(-.3893-16 ,	.1269+01)
.1162+01	(-.1192-15 ,	.1314+01)
.1194+01	(-.1623-15 ,	.1359+01)
.1225+01	(-.1659-15 ,	.1406+01)
.1257+01	(-.4241-15 ,	.1453+01)
.1288+01	(-.2170-15 ,	.1502+01)
.1319+01	(-.1334-15 ,	.1551+01)
.1351+01	(-.4557-16 ,	.1602+01)
.1382+01	(.0000 ,	.1655+01)
.1414+01	(-.4800-16 ,	.1708+01)
.1445+01	(-.9866-16 ,	.1763+01)
.1477+01	(-.4059-15 ,	.1820+01)
.1508+01	(-.3656-15 ,	.1878+01)
.1539+01	(-.2691-15 ,	.1938+01)

THETA	HBAR	
.1571+01	(-.2220-15 ,	.2000+01)
.1602+01	(-.2866-15 ,	.2064+01)
.1634+01	(-.2962-15 ,	.2130+01)
.1665+01	(-.4902-15 ,	.2198+01)
.1696+01	(-.6347-16 ,	.2269+01)
.1728+01	(-.6581-16 ,	.2342+01)
.1759+01	(.0000 ,	.2418+01)
.1791+01	(-.7100-16 ,	.2496+01)
.1822+01	(-.2217-15 ,	.2578+01)
.1854+01	(-.3850-15 ,	.2664+01)
.1885+01	(-.4820-15 ,	.2753+01)
.1916+01	(-.4197-15 ,	.2846+01)
.1948+01	(-.5271-15 ,	.2943+01)
.1979+01	(-.3683-15 ,	.3045+01)
.2011+01	(.0000 ,	.3151+01)
.2042+01	(-.1017-15 ,	.3264+01)
.2073+01	(-.4285-15 ,	.3382+01)
.2105+01	(-.3392-15 ,	.3506+01)
.2136+01	(-.4784-15 ,	.3638+01)
.2168+01	(-.1014-14 ,	.3777+01)
.2199+01	(-.6733-15 ,	.3925+01)
.2231+01	(-.7170-15 ,	.4083+01)
.2262+01	(-.6124-15 ,	.4250+01)
.2293+01	(-.1639-15 ,	.4430+01)
.2325+01	(-.1056-14 ,	.4622+01)
.2356+01	(-.9476-15 ,	.4828+01)
.2388+01	(-.8193-15 ,	.5051+01)
.2419+01	(-.6664-15 ,	.5293+01)
.2450+01	(-.3628-15 ,	.5555+01)
.2482+01	(-.5291-15 ,	.5842+01)
.2513+01	(-.1163-14 ,	.6155+01)
.2545+01	(-.1605-14 ,	.6501+01)
.2576+01	(-.1605-14 ,	.6884+01)
.2608+01	(-.5979-15 ,	.7311+01)
.2639+01	(-.1234-14 ,	.7789+01)
.2670+01	(.1273-15 ,	.8331+01)
.2702+01	(-.7291-15 ,	.8947+01)
.2733+01	(-.2531-14 ,	.9658+01)
.2765+01	(-.3755-14 ,	.1048+02)
.2796+01	(-.3169-14 ,	.1146+02)
.2827+01	(-.3403-14 ,	.1263+02)
.2859+01	(-.4544-14 ,	.1405+02)
.2890+01	(-.3313-14 ,	.1583+02)
.2922+01	(-.6051-14 ,	.1812+02)
.2953+01	(-.5876-14 ,	.2116+02)
.2985+01	(-.4932-14 ,	.2541+02)
.3016+01	(-.1452-13 ,	.3179+02)
.3047+01	(-.2013-13 ,	.4241+02)
.3079+01	(-.6066-13 ,	.6364+02)
.3110+01	(-.5632-12 ,	.1273+03)

TABLE AIII. - LINE PRINTOUTS OF ADAMS-MOULTON CORRECTORS - Continued

(c) k = 2, d = 3

--- ADAMS MOULTON K=2, D=3 ---

ALPHA(2) =	.1000000000000000+01	BETA(2) =	.4166666666666666+00
ALPHA(1) =	-.1000000000000000+01	BETA(1) =	.6666666666666665+00
ALPHA(0) =	.0000000000000000	BETA(0) =	-.8333333333333331-01

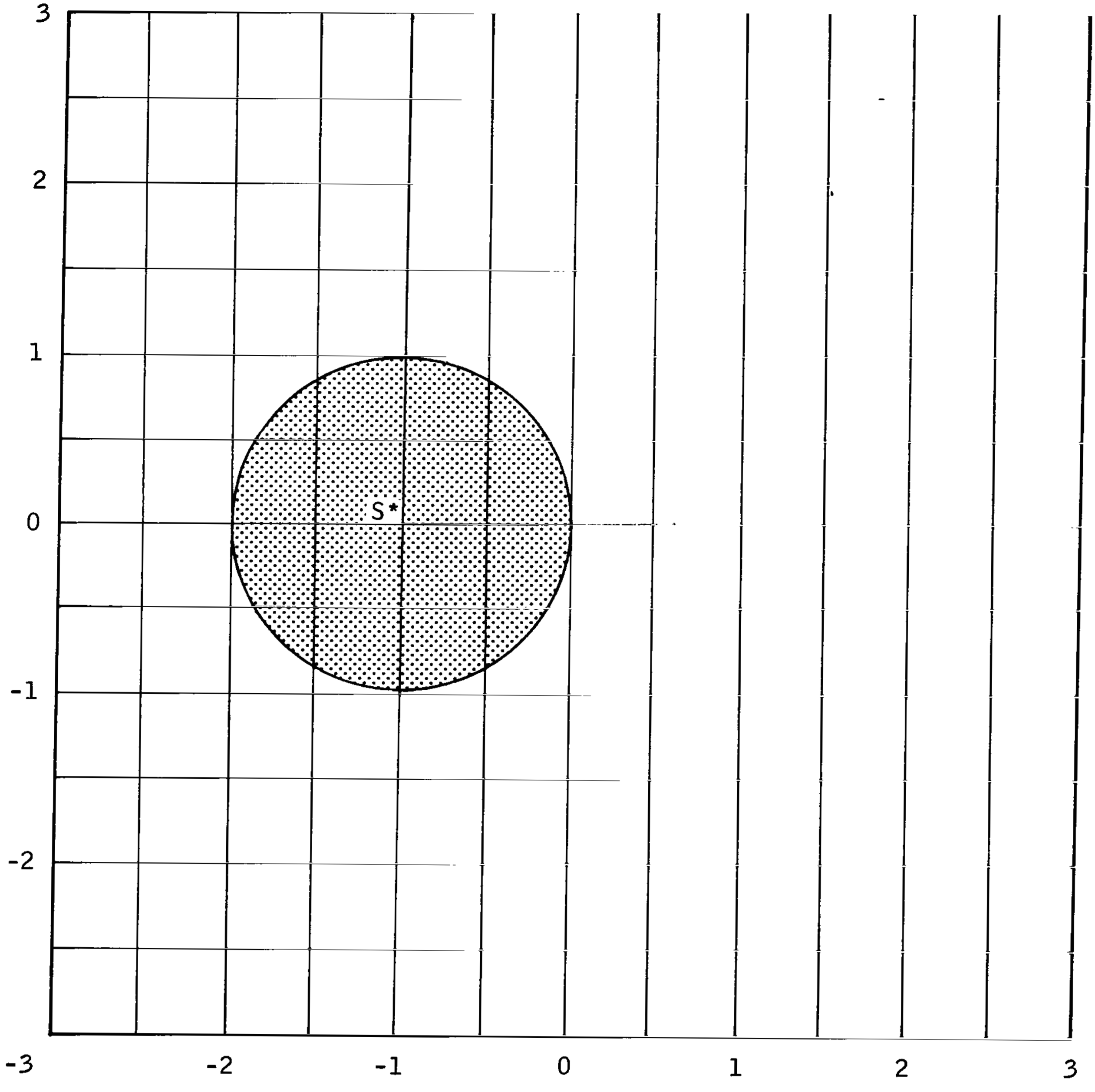
THETA	HBAR		THETA	HBAR	
.0000	(.0000 ,	-.0000)	.1571+01	(-.2400-00 ,	.1680+01)
.3142-01	(-.4058-07 ,	.3142-01)	.1602+01	(-.2606-00 ,	.1722+01)
.6283-01	(-.6492-06 ,	.6283-01)	.1634+01	(-.2827-00 ,	.1764+01)
.9425-01	(-.3285-05 ,	.9425-01)	.1665+01	(-.3063-00 ,	.1807+01)
.1257-00	(-.1038-04 ,	.1257-00)	.1696+01	(-.3316-00 ,	.1851+01)
.1571-00	(-.2532-04 ,	.1571-00)	.1728+01	(-.3586-00 ,	.1895+01)
.1885-00	(-.5245-04 ,	.1885-00)	.1759+01	(-.3876-00 ,	.1941+01)
.2199-00	(-.9706-04 ,	.2199-00)	.1791+01	(-.4185-00 ,	.1987+01)
.2513-00	(-.1654-03 ,	.2513-00)	.1822+01	(-.4517-00 ,	.2034+01)
.2827-00	(-.2646-03 ,	.2828-00)	.1854+01	(-.4872-00 ,	.2082+01)
.3142-00	(-.4027-03 ,	.3142-00)	.1885+01	(-.5251-00 ,	.2130+01)
.3456-00	(-.5886-03 ,	.3456-00)	.1916+01	(-.5658-00 ,	.2180+01)
.3770-00	(-.8322-03 ,	.3771-00)	.1948+01	(-.6093-00 ,	.2230+01)
.4084-00	(-.1144-02 ,	.4086-00)	.1979+01	(-.6559-00 ,	.2281+01)
.4398-00	(-.1536-02 ,	.4401-00)	.2011+01	(-.7058-00 ,	.2333+01)
.4712-00	(-.2020-02 ,	.4716-00)	.2042+01	(-.7593-00 ,	.2385+01)
.5027-00	(-.2609-02 ,	.5031-00)	.2073+01	(-.8166-00 ,	.2439+01)
.5341-00	(-.3318-02 ,	.5347-00)	.2105+01	(-.8781-00 ,	.2492+01)
.5655-00	(-.4161-02 ,	.5663-00)	.2136+01	(-.9441-00 ,	.2547+01)
.5969-00	(-.5153-02 ,	.5980-00)	.2168+01	(-.1015+01 ,	.2601+01)
.6283-00	(-.6312-02 ,	.6297-00)	.2199+01	(-.1091+01 ,	.2656+01)
.6597-00	(-.7654-02 ,	.6615-00)	.2231+01	(-.1172+01 ,	.2711+01)
.6912-00	(-.9196-02 ,	.6934-00)	.2262+01	(-.1260+01 ,	.2765+01)
.7226-00	(-.1096-01 ,	.7254-00)	.2293+01	(-.1354+01 ,	.2819+01)
.7540-00	(-.1296-01 ,	.7574-00)	.2325+01	(-.1455+01 ,	.2872+01)
.7854-00	(-.1522-01 ,	.7896-00)	.2356+01	(-.1563+01 ,	.2923+01)
.8168-00	(-.1776-01 ,	.8219-00)	.2388+01	(-.1679+01 ,	.2973+01)
.8482-00	(-.2061-01 ,	.8543-00)	.2419+01	(-.1804+01 ,	.3019+01)
.8796-00	(-.2378-01 ,	.8869-00)	.2450+01	(-.1938+01 ,	.3063+01)
.9111-00	(-.2730-01 ,	.9196-00)	.2482+01	(-.2082+01 ,	.3102+01)
.9425-00	(-.3120-01 ,	.9525-00)	.2513+01	(-.2236+01 ,	.3136+01)
.9739-00	(-.3550-01 ,	.9856-00)	.2545+01	(-.2400+01 ,	.3163+01)
.1005+01	(-.4023-01 ,	.1019+01)	.2576+01	(-.2575+01 ,	.3182+01)
.1037+01	(-.4541-01 ,	.1053+01)	.2608+01	(-.2761+01 ,	.3191+01)
.1068+01	(-.5108-01 ,	.1086+01)	.2639+01	(-.2959+01 ,	.3189+01)
.1100+01	(-.5727-01 ,	.1120+01)	.2670+01	(-.3167+01 ,	.3173+01)
.1131+01	(-.6401-01 ,	.1155+01)	.2702+01	(-.3386+01 ,	.3141+01)
.1162+01	(-.7134-01 ,	.1190+01)	.2733+01	(-.3615+01 ,	.3091+01)
.1194+01	(-.7929-01 ,	.1225+01)	.2765+01	(-.3852+01 ,	.3019+01)
.1225+01	(-.8790-01 ,	.1260+01)	.2796+01	(-.4095+01 ,	.2924+01)
.1257+01	(-.9722-01 ,	.1296+01)	.2827+01	(-.4342+01 ,	.2802+01)
.1288+01	(-.1073+00 ,	.1332+01)	.2859+01	(-.4589+01 ,	.2652+01)
.1319+01	(-.1181+00 ,	.1369+01)	.2890+01	(-.4832+01 ,	.2471+01)
.1351+01	(-.1298-00 ,	.1406+01)	.2922+01	(-.5067+01 ,	.2258+01)
.1382+01	(-.1424-00 ,	.1443+01)	.2953+01	(-.5288+01 ,	.2013+01)
.1414+01	(-.1559-00 ,	.1481+01)	.2985+01	(-.5488+01 ,	.1736+01)
.1445+01	(-.1705-00 ,	.1520+01)	.3016+01	(-.5663+01 ,	.1429+01)
.1477+01	(-.1861-00 ,	.1559+01)	.3047+01	(-.5806+01 ,	.1097+01)
.1508+01	(-.2028-00 ,	.1599+01)	.3079+01	(-.5912+01 ,	.7438-00)
.1539+01	(-.2208-00 ,	.1639+01)	.3110+01	(-.5978+01 ,	.3757-00)
.1571+01	(-.2400-00 ,	.1680+01)	.3142+01	(-.6000+01 ,	.1199-13)

TABLE AIII. - LINE PRINTOUTS OF ADAMS-MOULTON CORRECTORS - Concluded

(d) k = 3, d = 4

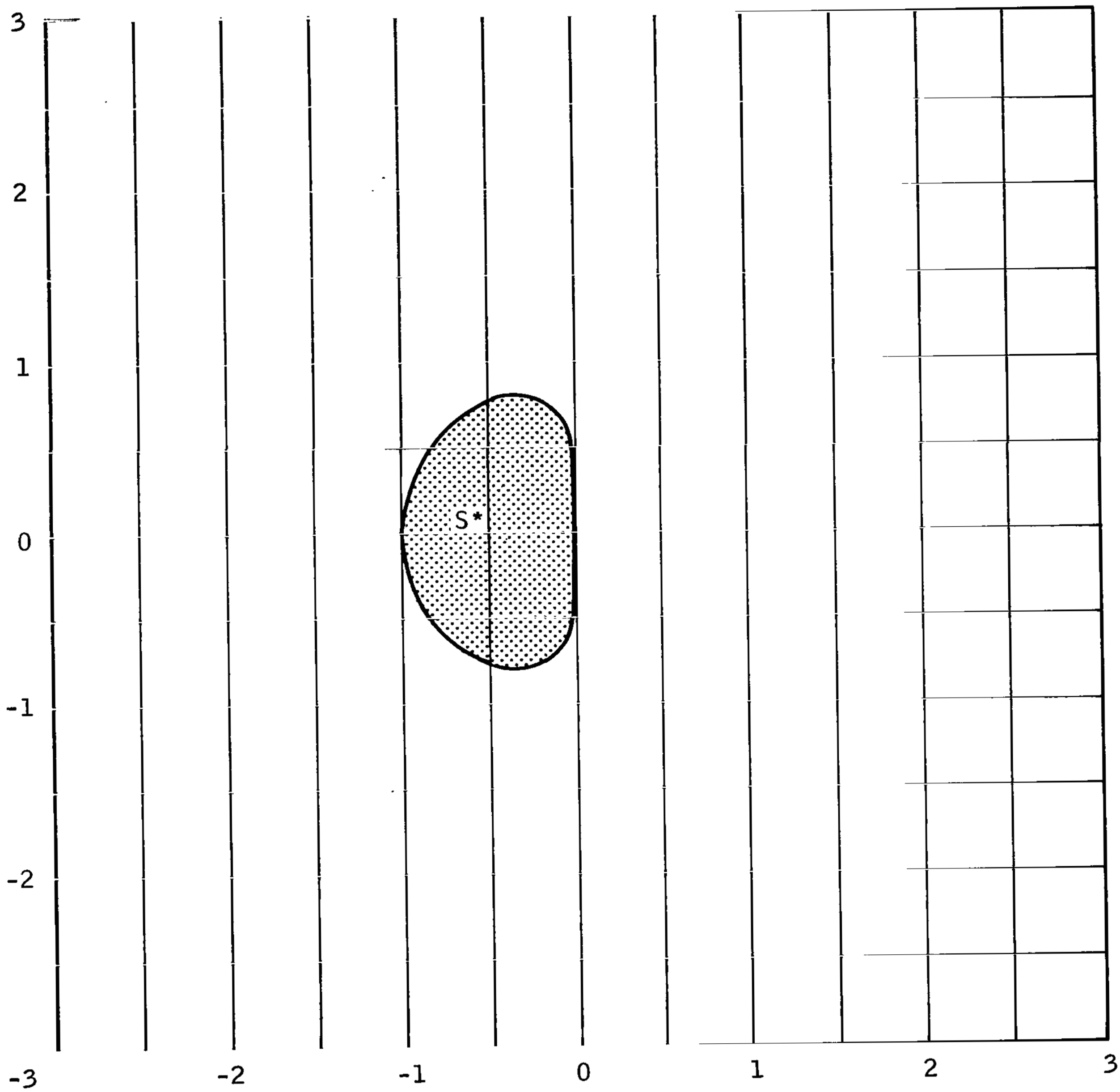
```
--- ADAMS MOULTON  K=3,  D=4        ---

ALPHA( 3) =  .1000000000000000+01          BETA( 3) =  .3750000000000000+00
ALPHA( 2) = -.1000000000000000+01          BETA( 2) =  .7916666666666665+00
ALPHA( 1) =  .0000000000000000             BETA( 1) = -.2083333333333332+00
ALPHA( 0) =  .0000000000000000             BETA( 0) =  .4166666666666666-01
```

THETA	HBAR		THETA	HBAR	
.0000	(.0000 ,	-.0000)	.1571+01	(-.1846-00 ,	.1477+01)
.3142-01	(-.2003-10 ,	.3142-01)	.1602+01	(-.2037-00 ,	.1502+01)
.6283-01	(-.1281-08 ,	.6283-01)	.1634+01	(-.2242-00 ,	.1527+01)
.9425-01	(-.1458-07 ,	.9425-01)	.1665+01	(-.2462-00 ,	.1552+01)
.1257-00	(-.8182-07 ,	.1257-00)	.1696+01	(-.2697-00 ,	.1576+01)
.1571-00	(-.3117-06 ,	.1571-00)	.1728+01	(-.2949-00 ,	.1600+01)
.1885-00	(-.9289-06 ,	.1885-00)	.1759+01	(-.3218-00 ,	.1623+01)
.2199-00	(-.2337-05 ,	.2199-00)	.1791+01	(-.3505-00 ,	.1647+01)
.2513-00	(-.5194-05 ,	.2513-00)	.1822+01	(-.3810-00 ,	.1669+01)
.2827-00	(-.1050-04 ,	.2827-00)	.1854+01	(-.4135-00 ,	.1692+01)
.3142-00	(-.1969-04 ,	.3141-00)	.1885+01	(-.4479-00 ,	.1714+01)
.3456-00	(-.3476-04 ,	.3455-00)	.1916+01	(-.4844-00 ,	.1735+01)
.3770-00	(-.5835-04 ,	.3768-00)	.1948+01	(-.5231-00 ,	.1755+01)
.4084-00	(-.9389-04 ,	.4081-00)	.1979+01	(-.5641-00 ,	.1775+01)
.4398-00	(-.1457-03 ,	.4394-00)	.2011+01	(-.6074-00 ,	.1793+01)
.4712-00	(-.2193-03 ,	.4707-00)	.2042+01	(-.6530-00 ,	.1811+01)
.5027-00	(-.3212-03 ,	.5019-00)	.2073+01	(-.7012-00 ,	.1828+01)
.5341-00	(-.4592-03 ,	.5330-00)	.2105+01	(-.7519-00 ,	.1843+01)
.5655-00	(-.6427-03 ,	.5641-00)	.2136+01	(-.8052-00 ,	.1857+01)
.5969-00	(-.8826-03 ,	.5952-00)	.2168+01	(-.8612-00 ,	.1869+01)
.6283-00	(-.1192-02 ,	.6261-00)	.2199+01	(-.9199-00 ,	.1880+01)
.6597-00	(-.1584-02 ,	.6570-00)	.2231+01	(-.9815-00 ,	.1888+01)
.6912-00	(-.2076-02 ,	.6877-00)	.2262+01	(-.1046+01 ,	.1894+01)
.7226-00	(-.2685-02 ,	.7183-00)	.2293+01	(-.1113+01 ,	.1897+01)
.7540-00	(-.3433-02 ,	.7488-00)	.2325+01	(-.1183+01 ,	.1898+01)
.7854-00	(-.4341-02 ,	.7792-00)	.2356+01	(-.1256+01 ,	.1896+01)
.8168-00	(-.5434-02 ,	.8095-00)	.2388+01	(-.1331+01 ,	.1890+01)
.8482-00	(-.6738-02 ,	.8395-00)	.2419+01	(-.1410+01 ,	.1880+01)
.8796-00	(-.8282-02 ,	.8695-00)	.2450+01	(-.1490+01 ,	.1867+01)
.9111-00	(-.1010-01 ,	.8992-00)	.2482+01	(-.1573+01 ,	.1848+01)
.9425-00	(-.1222-01 ,	.9288-00)	.2513+01	(-.1659+01 ,	.1825+01)
.9739-00	(-.1467-01 ,	.9581-00)	.2545+01	(-.1746+01 ,	.1797+01)
.1005+01	(-.1751-01 ,	.9873-00)	.2576+01	(-.1835+01 ,	.1762+01)
.1037+01	(-.2076-01 ,	.1016+01)	.2608+01	(-.1925+01 ,	.1722+01)
.1068+01	(-.2446-01 ,	.1045+01)	.2639+01	(-.2016+01 ,	.1676+01)
.1100+01	(-.2866-01 ,	.1074+01)	.2670+01	(-.2107+01 ,	.1622+01)
.1131+01	(-.3340-01 ,	.1102+01)	.2702+01	(-.2198+01 ,	.1562+01)
.1162+01	(-.3873-01 ,	.1130+01)	.2733+01	(-.2288+01 ,	.1494+01)
.1194+01	(-.4469-01 ,	.1158+01)	.2765+01	(-.2376+01 ,	.1418+01)
.1225+01	(-.5134-01 ,	.1186+01)	.2796+01	(-.2461+01 ,	.1335+01)
.1257+01	(-.5872-01 ,	.1213+01)	.2827+01	(-.2544+01 ,	.1244+01)
.1288+01	(-.6688-01 ,	.1241+01)	.2859+01	(-.2622+01 ,	.1146+01)
.1319+01	(-.7588-01 ,	.1268+01)	.2890+01	(-.2695+01 ,	.1040+01)
.1351+01	(-.8576-01 ,	.1295+01)	.2922+01	(-.2762+01 ,	.9278-00)
.1382+01	(-.9659-01 ,	.1321+01)	.2953+01	(-.2822+01 ,	.8087-00)
.1414+01	(-.1084+00 ,	.1348+01)	.2985+01	(-.2875+01 ,	.6836-00)
.1445+01	(-.1213+00 ,	.1374+01)	.3016+01	(-.2919+01 ,	.5535-00)
.1477+01	(-.1353-00 ,	.1400+01)	.3047+01	(-.2954+01 ,	.4190-00)
.1508+01	(-.1505-00 ,	.1426+01)	.3079+01	(-.2979+01 ,	.2812-00)
.1539+01	(-.1669-00 ,	.1452+01)	.3110+01	(-.2995+01 ,	.1412-00)
.1571+01	(-.1846-00 ,	.1477+01)	.3142+01	(-.3000+01 ,	.4673-14)



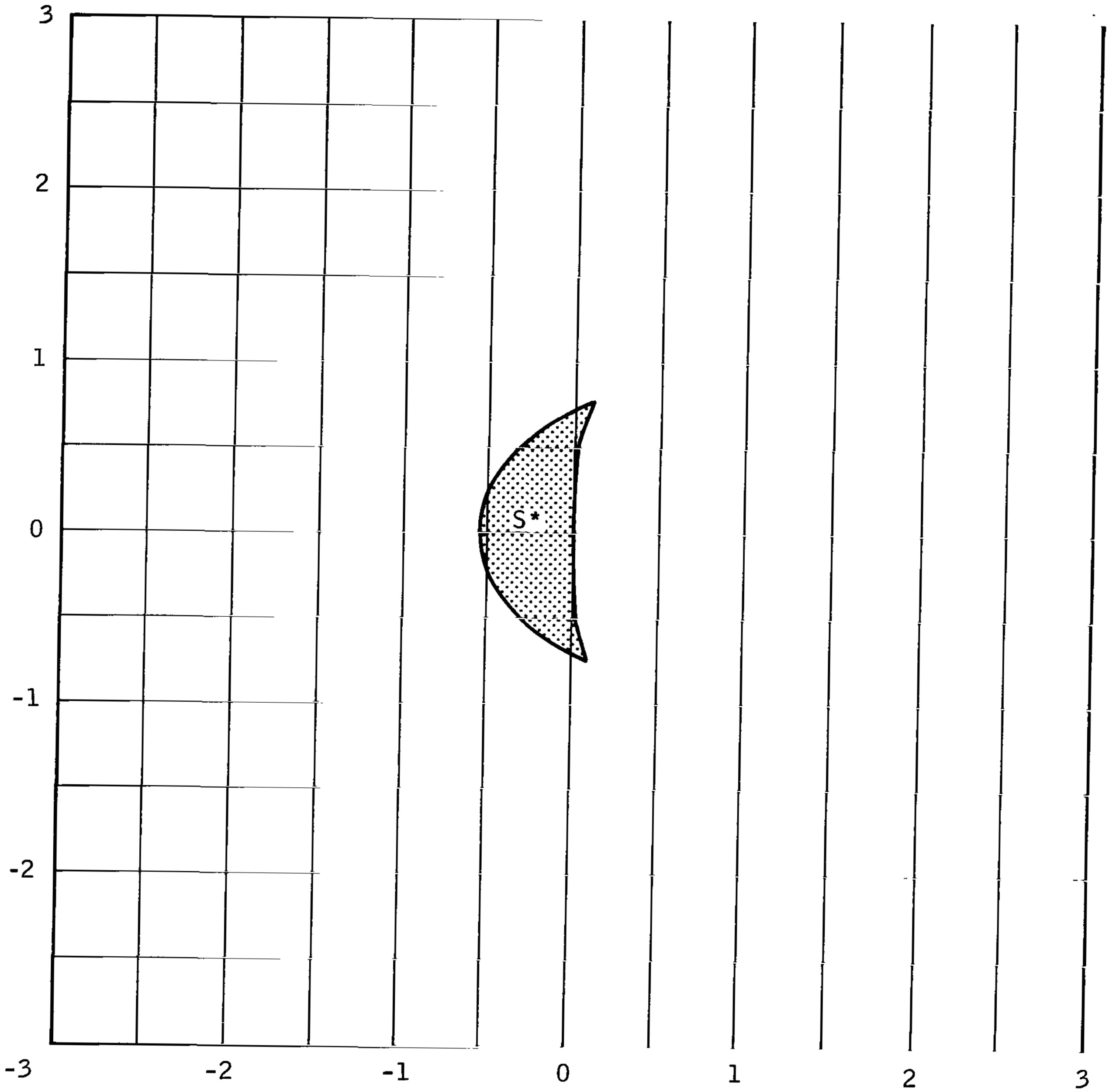
(a) k = 1, d = 1

Figure A1. - FORTRAN IV subroutine STBL1 plots of Adams-Bashforth predictors.



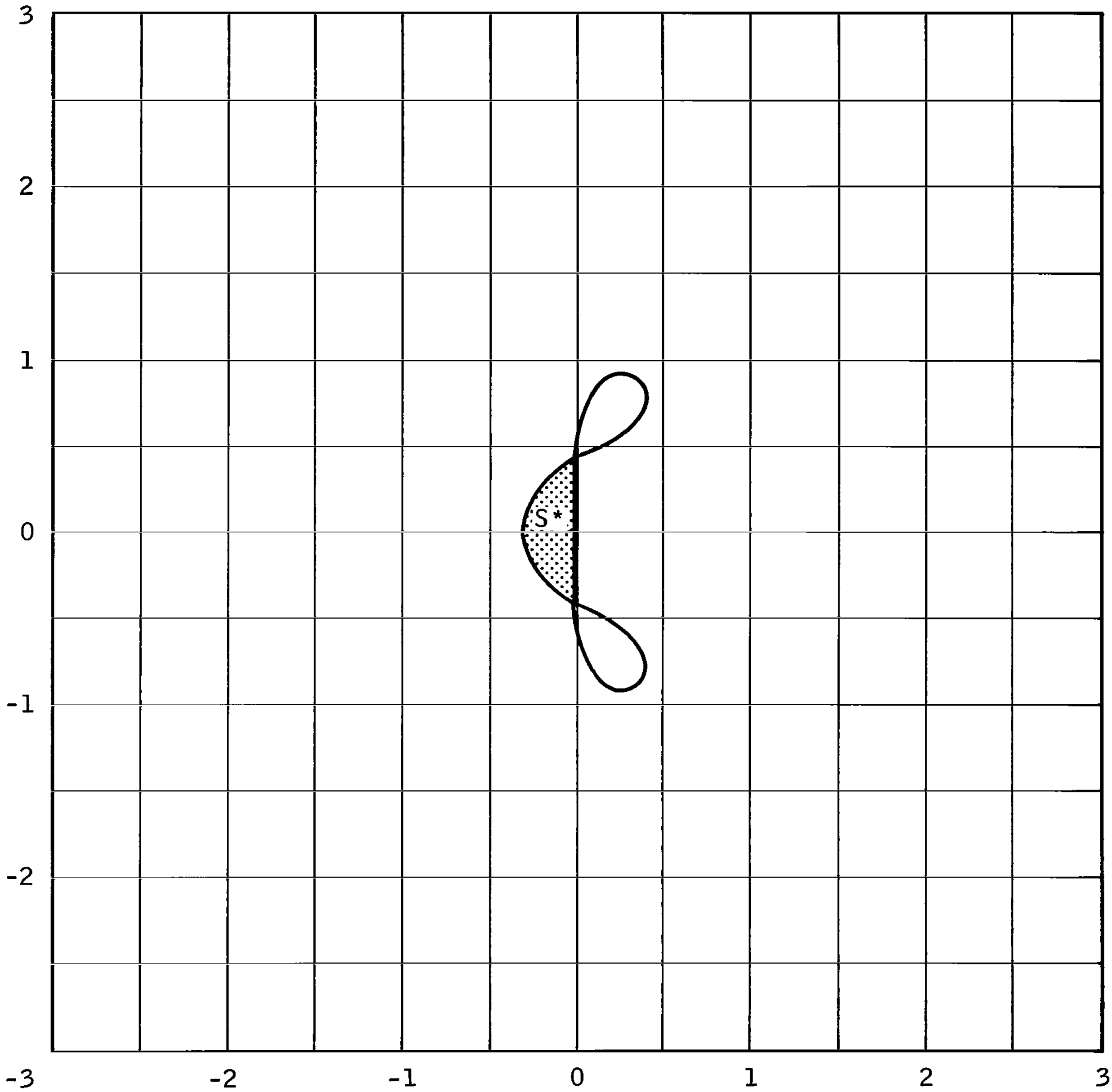
(b) k = 2, d = 2

Figure A1. - Continued.



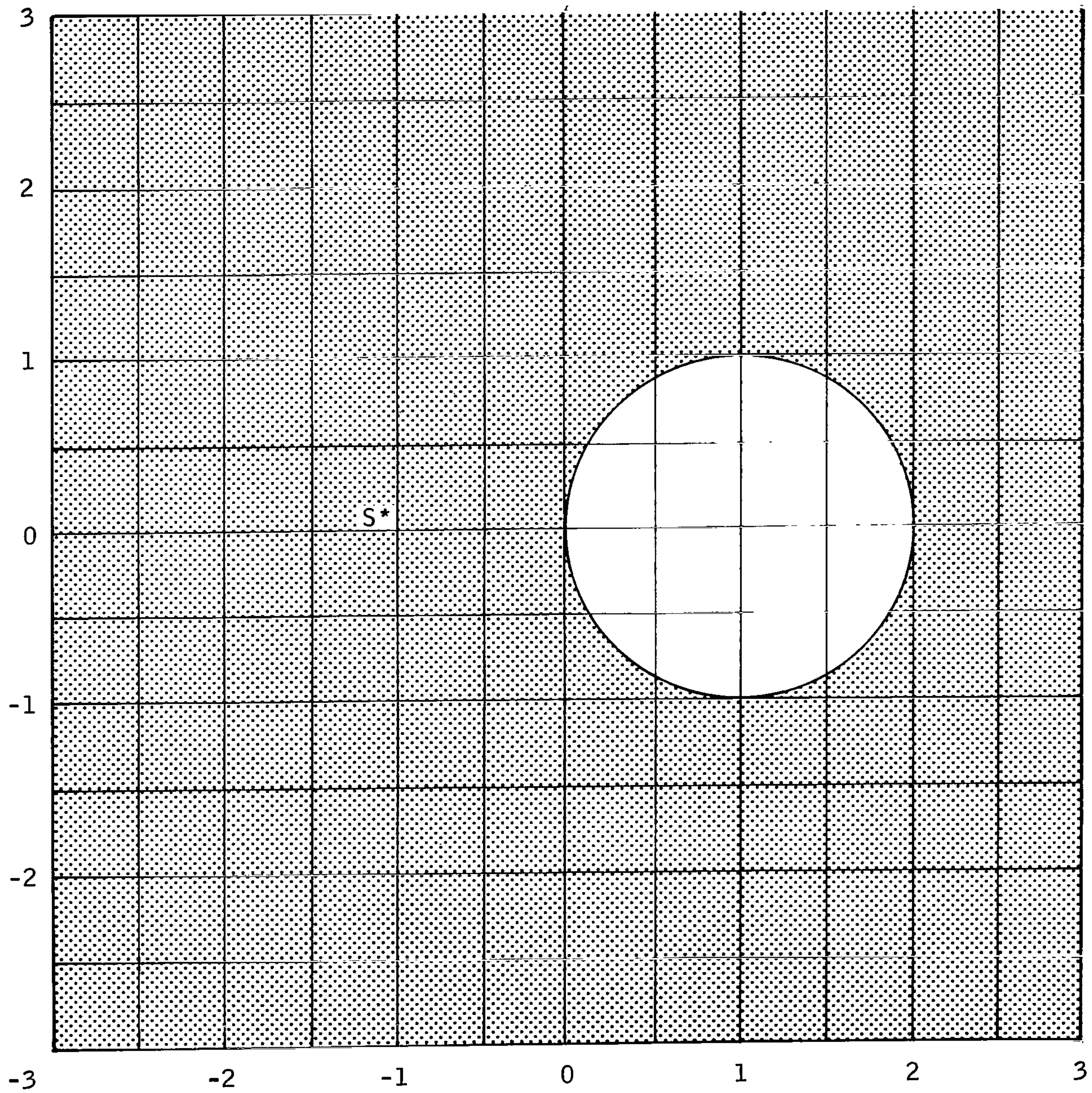
(c) k = 3, d = 3

Figure A1. - Continued.



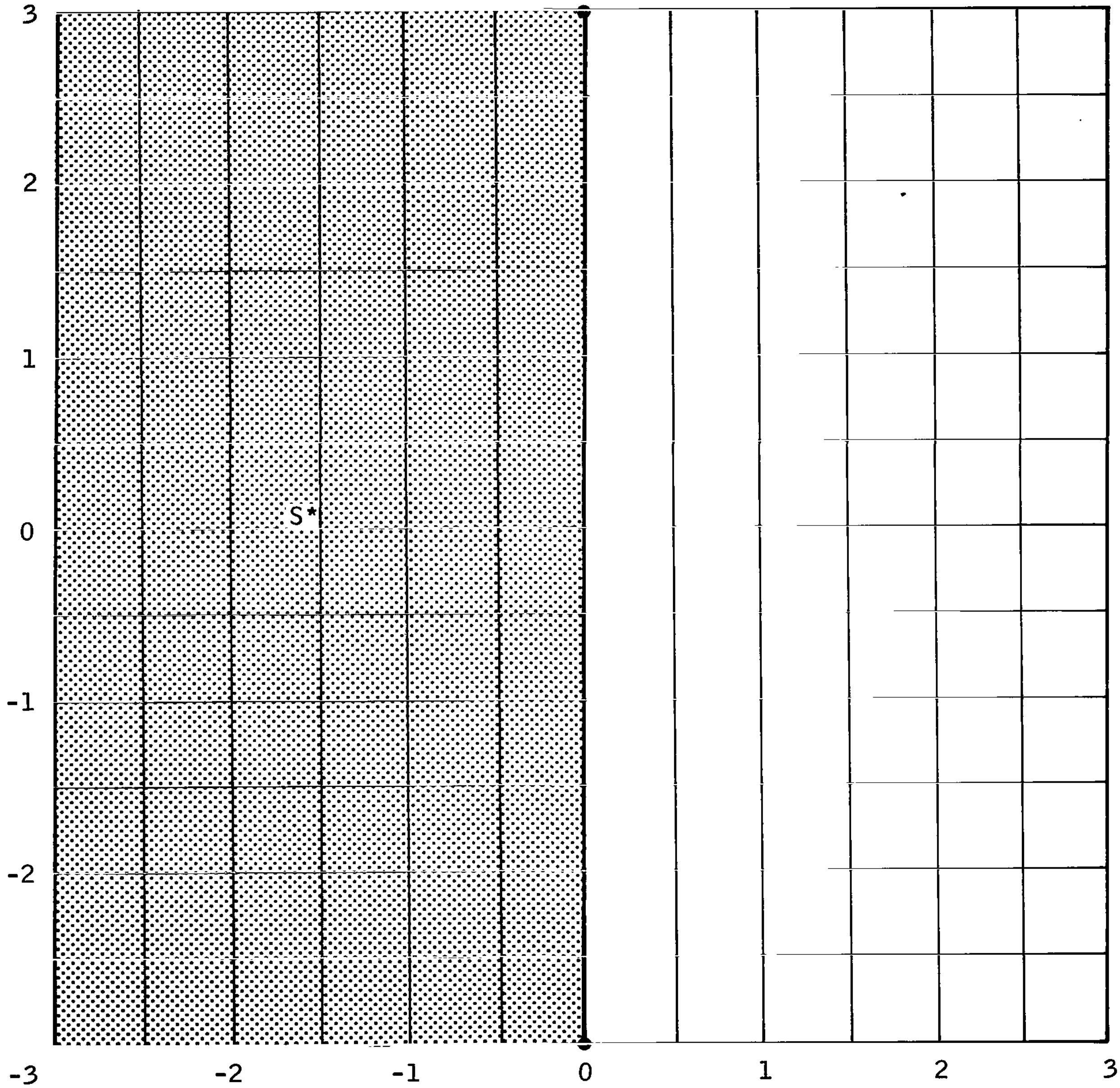
(d) k = 4, d = 4

Figure A1. - Concluded.



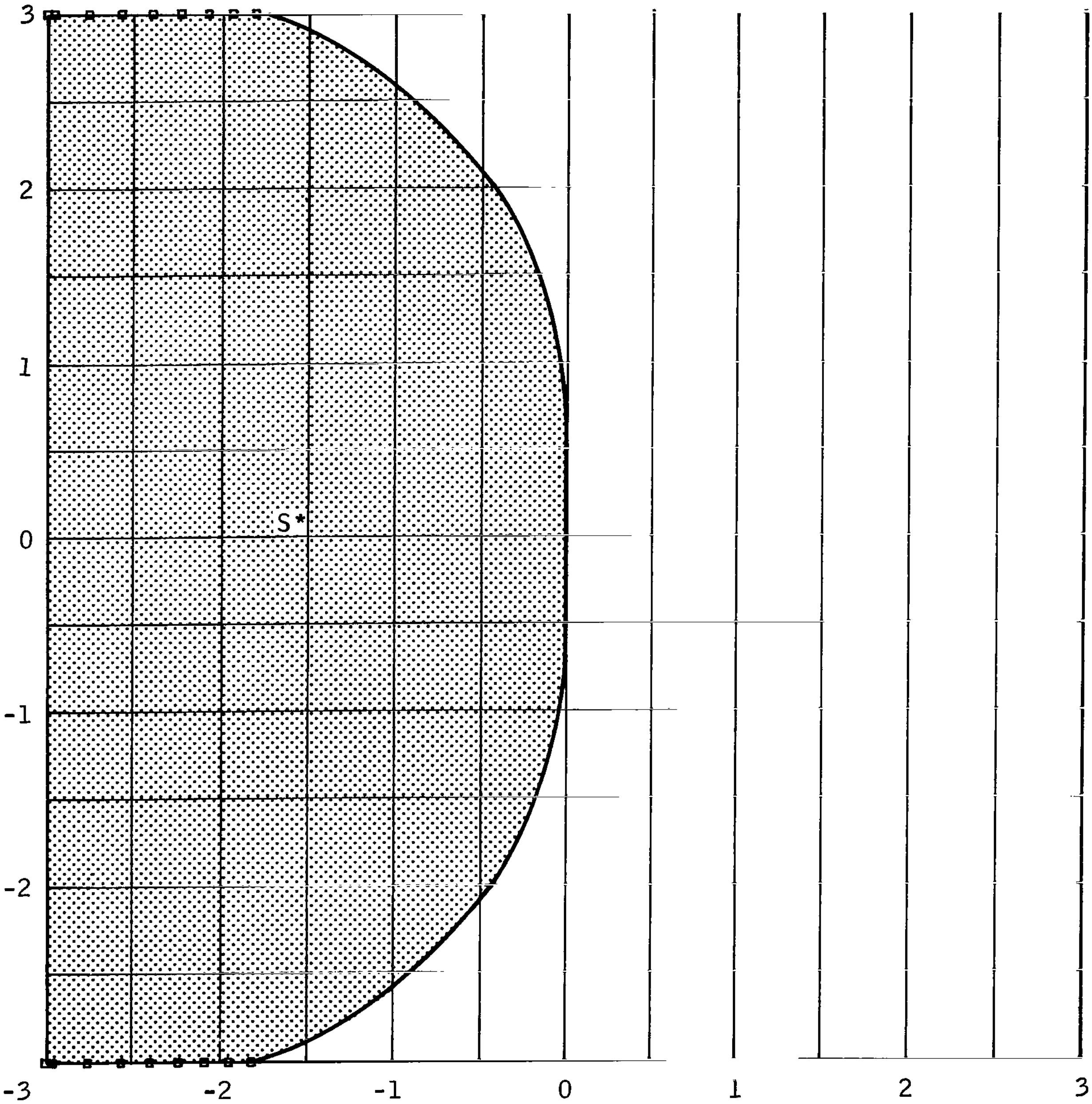
(a) k = 1, d = 1

Figure A2. - FORTRAN IV subroutine STBL1 plots of Adams-Moulton correctors.



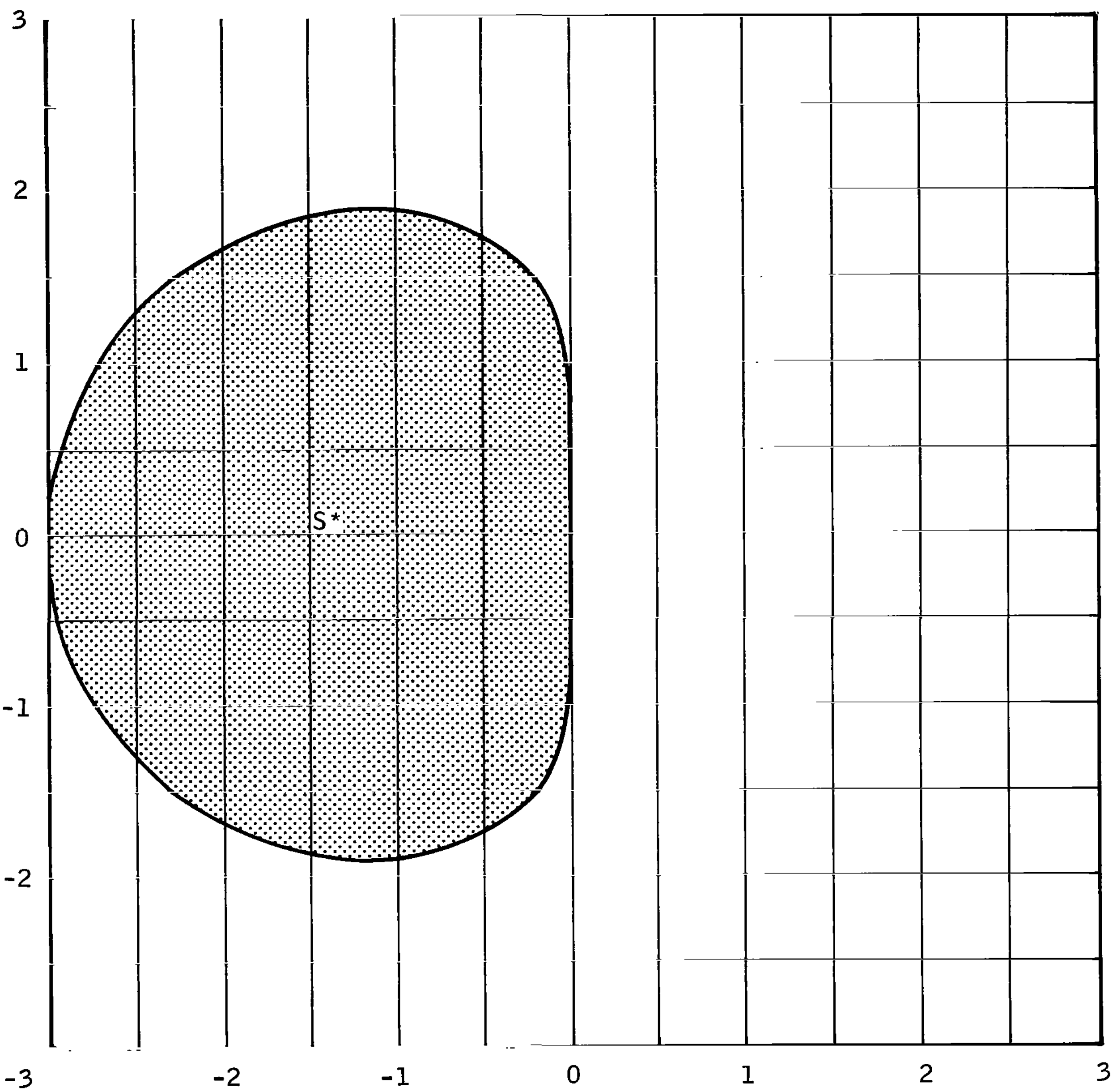
(b) k = 1, d = 2

Figure A2. - Continued.



(c) k = 2, d = 3

Figure A2. - Continued.



(d) k = 3, d = 4

Figure A2. - Concluded.

APPENDIX B

FORTRAN IV SUBROUTINE STBL2 FOR PREDICTOR-CORRECTOR MULTISTEP METHODS

The subroutine STBL2 computes the boundary of the region of stability for predictor-corrector multistep methods for which the corrector is applied only once.

In order to use the subroutine STBL2, the following calling sequence is necessary:

CALL STBL2 (AA1, BB1, AA2, BB2, K, N, TITLE)
Where the predictor-corrector multistep method is

Predictor: $\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \ldots + \alpha_0 y_n = h\left(\beta_{k-1} y'_{n+k-1} + \ldots + \beta_0 y'_n\right)$

Corrector: $\gamma_k y_{n+k} + \gamma_{k-1} y_{n+k-1} + \ldots + \gamma_0 y_n = h\left(\delta_k y'_{n+k} + \delta_{k-1} y'_{n+k-1} + \ldots + \delta_0 y'_n\right)$

and

AA1 is a double precision array with 25 locations such that $AA1(1) = \alpha_k$, $AA1(2) = \alpha_{k-1}$, ..., $AA1(k + 1) = \alpha_0$.

BB1 is a double precision array with 25 locations such that $BB1(1) = 0$, $BB1(2) = \beta_{k-1}$, ..., $BB1(k + 1) = \beta_0$.

AA2 is a double precision array with 25 locations such that $AA2(1) = \gamma_k$, $AA2(2) = \gamma_{k-1}$, ..., $AA2(k + 1) = \gamma_0$.

BB2 is a double precision array with 25 locations such that $BB2(1) = \delta_k$, $BB2(2) = \delta_{k-1}$, ..., $BB2(k + 1) = \delta_0$.

K is an integer giving the order of the method (use the largest order of the predictor or of the corrector).

N is an integer giving the number of divisions of the interval $(0, \Pi)$ to be used in computing the boundary of the region of stability; 90 divisions are usually sufficient.

TITLE is an array of 5 locations containing BCD information to be used as a title for the printed and plotted output. TITLE may be passed to the subroutine as 30H (any 30 BCD characters).

The subroutine CMPLX is used to perform the double precision complex arithmetic.

STBL2 will compute the boundary of the region of stability and print the results on the line printer if IØ = 6 or on the 4020 output if IØ = 17; it will also plot the results on the 4020 plotter.

Table BI is a listing of the FORTRAN IV subroutine STBL2. Subroutine STBL2 has been used to calculate the region of stability for the Bashforth-Moulton predictor-corrector pairs of degree 1 to 4. Table BII gives the coefficients of Bashforth-Moulton methods and the numerical results of STBL2, while figure B1 displays the results graphically. The coefficients for the methods are available from NASA Manned Spacecraft Center, Houston, Texas.

TABLE BI. - LISTING OF FORTRAN IV SUBROUTINE STBL 2 FOR PREDICTOR-CORRECTOR MULTISTEP METHODS

```
      DIMENSION AA(25),BB(25),AA1(25),BB1(25),AA2(25),BB2(25)
      DOUBLE PRECISION AA,BB,AA1,BB1,BB2,AA2
      AA1(1) = 1.0D0
      AA1(2) = -1.0D0
      AA2(1) = 1.0D0
      AA2(2) = -1.0D0
      DO 1001 J=3,25
      AA1(J) = 0.0D0
 1001 AA2(J) = 0.0D0
      BB(1) = 0.0D0
      BB(2) = 1.0D0
      CALL MOVE(BB,BB1)
      BB(1) = 1.0D0
      BB(2) = 0.0D0
      CALL MOVE(BB,BB2)
      KSTEP = 1
      CALL STBL2(AA1,BB1,AA2,BB2,KSTEP,90,
     1           30HBASHFORTH-MOULTON   K=1,   D=1    )
       BB(1) =   0.0D0
       BB(2) =   0.15000000000000000D1
       BB(3) = -0.50000000000000000D0
      CALL MOVE(BB,BB1)
       BB(1) = 0.5D0
       BB(2) = 0.5D0
      BB(3) = 0.0D0
      CALL MOVE(BB,BB2)
      KSTEP = 2
      CALL STBL2(AA1,BB1,AA2,BB2,KSTEP,90,
     1           30HBASHFORTH-MOULTON   K=2,   D=2    )
      BB(1) = 0.0D0
       BB(2) =   0.19166666666666667D1
       BB(3) = -0.13333333333333333D1
       BB(4) =   0.41666666666666667D0
      CALL MOVE(BB,BB1)
       BB(1) = 0.41666666666666667D-00
       BB(2) = 0.66666666666666667D-00
       BB(3) =-0.83333333333333333D-01
      BB(4) = 0.0D0
      CALL MOVE(BB,BB2)
      KSTEP = 3
      CALL STBL2(AA1,BB1,AA2,BB2,KSTEP,90,
     1           30HBASHFORTH-MOULTON   K=3,   D=3    )
      BB(1) = 0.0D0
       BB(2) =   0.22916666666666667D1
       BB(3) = -0.24583333333333333D1
       BB(4) =   0.15416666666666667D1
       BB(5) = -0.37500000000000000D0
      CALL MOVE(BB,BB1)
       BB(1) =   0.375D0
       BB(2) =   0.79166666666666667D-00
       BB(3) = -0.20833333333333333D-00
       BB(4) =   0.41666666666666667D-01
      BB(5) = 0.0D0
      CALL MOVE(BB,BB2)
      KSTEP = 4
      CALL STBL2(AA1,BB1,AA2,BB2,KSTEP,90,
     1           30HBASHFORTH-MOULTON   K=4,   D=4    )
      END
```

TABLE BI. - LISTING OF FORTRAN IV SUBROUTINE STBL2 FOR

PREDICTOR-CORRECTOR MULTISTEP METHODS - Continued

```
      SUBROUTINE MOVE(BB,BB1)
      DIMENSION BB(25),BB1(25)
      DOUBLE PRECISION BB,BB1
      DO 1010 I=1,25
 1010 BB1(I)=BB(I)
      RETURN
      END
```

```
      SUBROUTINE COMPLX (A, IA, B, IB, K, C, IC)
      INTEGER K
      DOUBLE PRECISION A, IA, B, IB, C, IC, TEMP, DSQRT
      DOUBLE PRECISION TEMPC
      GO TO (1, 2, 3), K
C     1 = MPY,   2 = DVD,   3 = SQT
    1 TEMPC = A * B - IA * IB
      IC = A * IB + IA * B
      C = TEMPC
      GO TO 100
   21 IC = 0.0D0
      C = 1.0D0
      GO TO 100
    2 CONTINUE
      TEMP = DABS(B**2 + IB**2)
      IF (TEMP .LT. 1.0D-20) GO TO 21
      TEMPC = (A * B + IA * IB) / TEMP
      IC = (IA * B - A * IB) / TEMP
      C = TEMPC
      GO TO 100
    3 TEMP = DSQRT(DABS(A**2 + IA**2))
      TEMPC = DSQRT(DABS(TEMP + A) / 2.0D0)
      MINUS = 1
      IF (IA .LT. 0.0D0) MINUS = -1
      IC = DSQRT(DABS(TEMP - A) / 2.0D0)
      C = TEMPC
      IC = MINUS*IC
      IF ((A .LT. 0.0D0) .AND. (IA .GT. 0.0D0)) GO TO 99
      GO TO 100
   99 CONTINUE
      C = -C
      IC = -IC
  100 RETURN
      END
```

TABLE BI. - LISTING OF FORTRAN IV SUBROUTINE STBL2 FOR

PREDICTOR-COLLECTOR MULTISTEP METHODS - Continued

```
      SUBROUTINE STBL2(AA1,BB1,AA2,BB2,K,N,TITLE)
      DIMENSION AA1(25),BB1(25),AA2(25),BB2(25),H1(500),H2(500),
     1          H3(500),H4(500),TITLE(5)
      DIMENSION TH(500)
      DIMENSION BCDX(12), BCDY(12)
      DATA BCDX/12*6H      /
      DATA BCDY/12*6H      /
      DOUBLE PRECISION AA1,BB1,AA2,BB2,AR,AI,BR,BI,CR,CI,DR,DI,DCOS,
     1                 DSIN,ACR,ACI,BBR,BBI,DSR,DSI,HR,HI,DL
      DOUBLE PRECISION THETA, DEL
      IO = 6
      IO = 17
      K1=K+1
      IC=0
      NC=0
      DEL=3.14159265358979300/N
      THETA=0.0D0
 1020 IF (NC .GT. N) GO TO 1030
      THETA = NC*DEL
      AR=0.0D0
      AI=0.0D0
      BR=0.0D0
      BI=0.0D0
      CR=0.0D0
      CI=0.0D0
      DO 1010 I=1,K1
      L=K-I+1
      DL=L
      IF (I .EQ. 1) GO TO 1021
      AR=AR+BB1(I)*DCOS(DL*THETA)
      AI=AI+BB1(I)*DSIN(DL*THETA)
      BR = BR+(BB2(I)-BB2(1)*AA1(I))*DCOS(DL*THETA)
      BI = BI+(BB2(I)-BB2(1)*AA1(I))*DSIN(DL*THETA)
 1021 CONTINUE
      CR=CR-AA2(I)*DCOS(DL*THETA)
      CI=CI-AA2(I)*DSIN(DL*THETA)
 1010 CONTINUE
      AR=AR*BB2(1)
      AI=AI*BB2(1)
      CALL COMPLX(4*AR,4*AI,CR,CI,1,ACR,ACI)
      CALL COMPLX(BR,BI,BR,BI,1,BBR,BBI)
      DR=BBR-ACR
      DI=BBI-ACI
      CALL COMPLX(DR,DI,DR,DI,3,DSR,DSI)
      DR=-BR+DSR
      DI=-BI+DSI
      CALL COMPLX(DR,DI,2*AR,2*AI,2,HR,HI)
      IC = IC + 1
      H1(IC) = HR
      H2(IC) = HI
      TH(IC) = THETA
      DR = -BR-DSR
      DI = -BI-DSI
      CALL COMPLX(DR,DI,2*AR,2*AI,2,HR,HI)
      IF (IC .EQ. 1) GO TO 1140
      DIF1 = (H1(IC-1)-HR)**2+(H2(IC-1)-HI)**2
      DIF3 = (H3(IC-1)-HR)**2+(H4(IC-1)-HI)**2
      DIF2=(H1(IC-1)-H1(IC))**2+(H2(IC-1)-H2(IC))**2
      DIF4=(H3(IC-1)-H1(IC))**2+(H4(IC-1)-H2(IC))**2
      IF ((DIF3 .LT. DIF1) .OR. (DIF2 .LT. DIF4))GO TO 1140
```

TABLE BI. - LISTING OF FORTRAN IV SUBROUTINE STBL2 FOR PREDICTOR-COLLECTOR MULTISTEP METHODS - Concluded

```
      H3(IC) = H1(IC)
      H4(IC) = H2(IC)
      H1(IC) = HR
      H2(IC) = HI
      GO TO 1150
 1140 CONTINUE
      H3(IC)=HR
      H4(IC)=HI
 1150 CONTINUE
 1040 NC=NC+1
      GO TO 1020
 1030 CONTINUE
 6001 FORMAT(1H1,40X,4H--- ,5A6,4H ---,///)
      WRITE(10,6001) (TITLE(J),J=1,5)
      DO 1060 J=1,K1
      L=K-J+1
 1060 WRITE(IO,6002) L,AA1(J),L,BB1(J)
 6002 FORMAT(1H ,15X,6HALPHA(,I2,4H) = ,D21.16,20X,
     1            6H BETA(,I2,4H) = ,D21.16)
 6005 FORMAT(///)
      WRITE(IO,6005)
      DO 1061 J=1,K1
      L=K-J+1
 1061 WRITE(10,6004) L,AA2(J),L,BB2(J)
 6004 FORMAT(1H ,15X,6HGAMMA(,I2,4H) = ,D21.16,20X,
     1             6HDELTA(,I2,4H) = , D21.16)
 6006 FORMAT(1H ,16X,5HTHETA,17X,5HHBAR1,45X,5HHBAR2)
      WRITE(10,6005)
      WRITE(10,6006)
 6003 FORMAT(1H ,13X,E10.4,5X,1H(,E10.4,3H , ,E10.4,1H),10X,
     1             15X,1H(,E10.4,3H , ,E10.4,1H))
      DO 1070 I=1,IC
      WRITE(10,6003) (TH(I),H1(I),H2(I),H3(I),H4(I))
 1070 CONTINUE
      CALL DMPBUF
      DO 1180 I=1,5
 1180 BCDX(I) = TITLE(I)
      CALL QUIKML(-1,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H1,H2)
      CALL QUIKML( 0,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H3,H4)
      DO 1190 I=1,IC
      H2(I)=-H2(I)
 1190 H4(I) = -H4(I)
      CALL QUIKML( 0,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H1,H2)
      CALL QUIKML( 0,-3.0,3.0,-3.0,3.0,1H ,BCDX,BCDY,IC,H3,H4)
      RETURN
      END
```

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS

(a) k = 1, d = 1

```
--- BASHFORTH-MOULTON  K=1,  D=1   ---

ALPHA( 1) =  .1000000000000000+01        BETA( 1) □  .0000000000000000
ALPHA( 0) = -.1000000000000000+01        BETA( 0) =  .1000000000000000+01

GAMMA( 1) =  .1000000000000000+01       DELTA( 1) =  .1000000000000000+01
GAMMA( 0) = -.1000000000000000+01       DELTA( 0) =  .0000000000000000
```

THETA	HBAR1	HBAR2
.0000	(.0000 , -.0000)	(-.1000+01 , -.0000)
.3491-01	(.6055-03 , .3486-01)	(-.1001+01 , -.3486-01)
.6981-01	(.2378-02 , .6943-01)	(-.1002+01 , -.6943-01)
.1047+00	(.5197-02 , .1035+00)	(-.1005+01 , -.1035+00)
.1396-00	(.8888-02 , .1367-00)	(-.1009+01 , -.1367-00)
.1745-00	(.1325-01 , .1692-00)	(-.1013+01 , -.1692-00)
.2094-00	(.1808-01 , .2007-00)	(-.1018+01 , -.2007-00)
.2443-00	(.2321-01 , .2312-00)	(-.1023+01 , -.2312-00)
.2793-00	(.2846-01 , .2608-00)	(-.1028+01 , -.2608-00)
.3142-00	(.3372-01 , .2895-00)	(-.1034+01 , -.2895-00)
.3491-00	(.3889-01 , .3173-00)	(-.1039+01 , -.3173-00)
.3840-00	(.4387-01 , .3444-00)	(-.1044+01 , -.3444-00)
.4189-00	(.4860-01 , .3707-00)	(-.1049+01 , -.3707-00)
.4538-00	(.5305-01 , .3963-00)	(-.1053+01 , -.3963-00)
.4887-00	(.5717-01 , .4213-00)	(-.1057+01 , -.4213-00)
.5236-00	(.6094-01 , .4457-00)	(-.1061+01 , -.4457-00)
.5585-00	(.6434-01 , .4695-00)	(-.1064+01 , -.4695-00)
.5934-00	(.6736-01 , .4928-00)	(-.1067+01 , -.4928-00)
.6283-00	(.6998-01 , .5156-00)	(-.1070+01 , -.5156-00)
.6632-00	(.7221-01 , .5380-00)	(-.1072+01 , -.5380-00)
.6981-00	(.7403-01 , .5599-00)	(-.1074+01 , -.5599-00)
.7330-00	(.7546-01 , .5814-00)	(-.1075+01 , -.5814-00)
.7679-00	(.7649-01 , .6025-00)	(-.1076+01 , -.6025-00)
.8029-00	(.7712-01 , .6232-00)	(-.1077+01 , -.6232-00)
.8378-00	(.7735-01 , .6436-00)	(-.1077+01 , -.6436-00)
.8727-00	(.7719-01 , .6636-00)	(-.1077+01 , -.6636-00)
.9076-00	(.7665-01 , .6833-00)	(-.1077+01 , -.6833-00)
.9425-00	(.7572-01 , .7026-00)	(-.1076+01 , -.7026-00)
.9774-00	(.7441-01 , .7216-00)	(-.1074+01 , -.7216-00)
.1012+01	(.7274-01 , .7403-00)	(-.1073+01 , -.7403-00)
.1047+01	(.7070-01 , .7587-00)	(-.1071+01 , -.7587-00)
.1082+01	(.6829-01 , .7768-00)	(-.1068+01 , -.7768-00)
.1117+01	(.6553-01 , .7946-00)	(-.1066+01 , -.7946-00)
.1152+01	(.6243-01 , .8121-00)	(-.1062+01 , -.8121-00)
.1187+01	(.5897-01 , .8294-00)	(-.1059+01 , -.8294-00)
.1222+01	(.5518-01 , .8463-00)	(-.1055+01 , -.8463-00)
.1257+01	(.5106-01 , .8629-00)	(-.1051+01 , -.8629-00)
.1292+01	(.4661-01 , .8793-00)	(-.1047+01 , -.8793-00)
.1326+01	(.4185-01 , .8954-00)	(-.1042+01 , -.8954-00)
.1361+01	(.3676-01 , .9112-00)	(-.1037+01 , -.9112-00)
.1396+01	(.3137-01 , .9267-00)	(-.1031+01 , -.9267-00)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Continued

(a) k = 1, d = 1

.1431+01	(.2568-01 ,	.9419-00)	(-.1026+01 ,	-.9419-00)
.1466+01	(.1969-01 ,	.9568-00)	(-.1020+01 ,	-.9568-00)
.1501+01	(.1341-01 ,	.9715-00)	(-.1013+01 ,	-.9715-00)
.1536+01	(.6844-02 ,	.9859-00)	(-.1007+01 ,	-.9859-00)
.1571+01	(.0000 ,	.1000+01)	(-.1000+01 ,	-.1000+01)
.1606+01	(-.7117-02 ,	.1014+01)	(-.9929-00 ,	-.1014+01)
.1641+01	(-.1450-01 ,	.1027+01)	(-.9855-00 ,	-.1027+01)
.1676+01	(-.2215-01 ,	.1041+01)	(-.9779-00 ,	-.1041+01)
.1710+01	(-.3005-01 ,	.1054+01)	(-.9700-00 ,	-.1054+01)
.1745+01	(-.3820-01 ,	.1066+01)	(-.9618-00 ,	-.1066+01)
.1780+01	(-.4659-01 ,	.1079+01)	(-.9534-00 ,	-.1079+01)
.1815+01	(-.5522-01 ,	.1091+01)	(-.9448-00 ,	-.1091+01)
.1850+01	(-.6408-01 ,	.1103+01)	(-.9359-00 ,	-.1103+01)
.1885+01	(-.7317-01 ,	.1114+01)	(-.9268-00 ,	-.1114+01)
.1920+01	(-.8248-01 ,	.1125+01)	(-.9175-00 ,	-.1125+01)
.1955+01	(-.9200-01 ,	.1136+01)	(-.9080-00 ,	-.1136+01)
.1990+01	(-.1017+00 ,	.1147+01)	(-.8983-00 ,	-.1147+01)
.2025+01	(-.1117+00 ,	.1157+01)	(-.8883-00 ,	-.1157+01)
.2059+01	(-.1218+00 ,	.1167+01)	(-.8782-00 ,	-.1167+01)
.2094+01	(-.1321-00 ,	.1177+01)	(-.8679-00 ,	-.1177+01)
.2129+01	(-.1426-00 ,	.1186+01)	(-.8574-00 ,	-.1186+01)
.2164+01	(-.1533-00 ,	.1196+01)	(-.8467-00 ,	-.1196+01)
.2199+01	(-.1641-00 ,	.1204+01)	(-.8359-00 ,	-.1204+01)
.2234+01	(-.1752-00 ,	.1213+01)	(-.8248-00 ,	-.1213+01)
.2269+01	(-.1863-00 ,	.1221+01)	(-.8137-00 ,	-.1221+01)
.2304+01	(-.1977-00 ,	.1229+01)	(-.8023-00 ,	-.1229+01)
.2339+01	(-.2092-00 ,	.1237+01)	(-.7908-00 ,	-.1237+01)
.2374+01	(-.2208-00 ,	.1244+01)	(-.7792-00 ,	-.1244+01)
.2409+01	(-.2325-00 ,	.1251+01)	(-.7675-00 ,	-.1251+01)
.2443+01	(-.2444-00 ,	.1258+01)	(-.7556-00 ,	-.1258+01)
.2478+01	(-.2564-00 ,	.1264+01)	(-.7436-00 ,	-.1264+01)
.2513+01	(-.2686-00 ,	.1270+01)	(-.7314-00 ,	-.1270+01)
.2548+01	(-.2808-00 ,	.1276+01)	(-.7192-00 ,	-.1276+01)
.2583+01	(-.2932-00 ,	.1281+01)	(-.7068-00 ,	-.1281+01)
.2618+01	(-.3056-00 ,	.1286+01)	(-.6944-00 ,	-.1286+01)
.2653+01	(-.3181-00 ,	.1291+01)	(-.6819-00 ,	-.1291+01)
.2688+01	(-.3308-00 ,	.1295+01)	(-.6692-00 ,	-.1295+01)
.2723+01	(-.3435-00 ,	.1299+01)	(-.6565-00 ,	-.1299+01)
.2758+01	(-.3563-00 ,	.1303+01)	(-.6437-00 ,	-.1303+01)
.2793+01	(-.3691-00 ,	.1306+01)	(-.6309-00 ,	-.1306+01)
.2827+01	(-.3820-00 ,	.1310+01)	(-.6180-00 ,	-.1310+01)
.2862+01	(-.3950-00 ,	.1312+01)	(-.6050-00 ,	-.1312+01)
.2897+01	(-.4080-00 ,	.1315+01)	(-.5920-00 ,	-.1315+01)
.2932+01	(-.4211-00 ,	.1317+01)	(-.5789-00 ,	-.1317+01)
.2967+01	(-.4342-00 ,	.1319+01)	(-.5658-00 ,	-.1319+01)
.3002+01	(-.4473-00 ,	.1320+01)	(-.5527-00 ,	-.1320+01)
.3037+01	(-.4604-00 ,	.1321+01)	(-.5396-00 ,	-.1321+01)
.3072+01	(-.4736-00 ,	.1322+01)	(-.5264-00 ,	-.1322+01)
.3107+01	(-.4868-00 ,	.1323+01)	(-.5132-00 ,	-.1323+01)
.3142+01	(-.5000-00 ,	.1323+01)	(-.5000-00 ,	-.1323+01)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Continued

(b) k = 2, d = 2

--- BASHFORTH-MOULTON K=2, D=2 ---

```
ALPHA( 2) =  .1000000000000000+01          BETA( 2) =  .0000000000000000
ALPHA( 1) = -.1000000000000000+01          BETA( 1) =  .1500000000000000+01
ALPHA( 0) =  .0000000000000000             BETA( 0) = -.5000000000000000+00

GAMMA( 2) =  .1000000000000000+01          DELTA( 2) =  .5000000000000000+00
GAMMA( 1) = -.1000000000000000+01          DELTA( 1) =  .5000000000000000+00
GAMMA( 0) =  .0000000000000000             DELTA( 0) =  .0000000000000000
```

THETA	HBAR1		HBAR2	
.0000	(.0000 ,	-.0000)	(-.2000+01 ,	-.0000)
.3491-01	(.3710-06 ,	.3491-01)	(-.1999+01 ,	-.4248-04)
.6981-01	(.5926-05 ,	.6984-01)	(-.1995+01 ,	-.3386-03)
.1047+00	(.2992-04 ,	.1048+00)	(-.1989+01 ,	-.1136-02)
.1396-00	(.9421-04 ,	.1398-00)	(-.1981+01 ,	-.2670-02)
.1745-00	(.2288-03 ,	.1749-00)	(-.1971+01 ,	-.5157-02)
.2094-00	(.4712-03 ,	.2101-00)	(-.1958+01 ,	-.8793-02)
.2443-00	(.8656-03 ,	.2453-00)	(-.1944+01 ,	-.1374-01)
.2793-00	(.1461-02 ,	.2806-00)	(-.1928+01 ,	-.2015-01)
.3142-00	(.2311-02 ,	.3160-00)	(-.1911+01 ,	-.2810-01)
.3491-00	(.3468-02 ,	.3513-00)	(-.1893+01 ,	-.3767-01)
.3840-00	(.4984-02 ,	.3866-00)	(-.1874+01 ,	-.4889-01)
.4189-00	(.6908-02 ,	.4218-00)	(-.1854+01 ,	-.6174-01)
.4538-00	(.9278-02 ,	.4568-00)	(-.1834+01 ,	-.7618-01)
.4887-00	(.1212-01 ,	.4915-00)	(-.1813+01 ,	-.9215-01)
.5236-00	(.1546-01 ,	.5259-00)	(-.1792+01 ,	-.1095+00)
.5585-00	(.1928-01 ,	.5598-00)	(-.1772+01 ,	-.1282-00)
.5934-00	(.2357-01 ,	.5931-00)	(-.1751+01 ,	-.1481-00)
.6283-00	(.2830-01 ,	.6258-00)	(-.1731+01 ,	-.1689-00)
.6632-00	(.3342-01 ,	.6578-00)	(-.1712+01 ,	-.1906-00)
.6981-00	(.3886-01 ,	.6889-00)	(-.1692+01 ,	-.2131-00)
.7330-00	(.4455-01 ,	.7192-00)	(-.1674+01 ,	-.2361-00)
.7679-00	(.5041-01 ,	.7485-00)	(-.1655+01 ,	-.2596-00)
.8029-00	(.5636-01 ,	.7768-00)	(-.1638+01 ,	-.2834-00)
.8378-00	(.6232-01 ,	.8042-00)	(-.1620+01 ,	-.3075-00)
.8727-00	(.6819-01 ,	.8305-00)	(-.1603+01 ,	-.3317-00)
.9076-00	(.7391-01 ,	.8558-00)	(-.1586+01 ,	-.3560-00)
.9425-00	(.7940-01 ,	.8801-00)	(-.1570+01 ,	-.3802-00)
.9774-00	(.8458-01 ,	.9035-00)	(-.1554+01 ,	-.4044-00)
.1012+01	(.8942-01 ,	.9258-00)	(-.1538+01 ,	-.4285-00)
.1047+01	(.9384-01 ,	.9473-00)	(-.1522+01 ,	-.4524-00)
.1082+01	(.9780-01 ,	.9678-00)	(-.1507+01 ,	-.4762-00)
.1117+01	(.1013+00 ,	.9875-00)	(-.1492+01 ,	-.4997-00)
.1152+01	(.1042+00 ,	.1006+01)	(-.1476+01 ,	-.5230-00)
.1187+01	(.1066+00 ,	.1024+01)	(-.1461+01 ,	-.5460-00)
.1222+01	(.1084+00 ,	.1042+01)	(-.1446+01 ,	-.5688-00)
.1257+01	(.1097+00 ,	.1058+01)	(-.1431+01 ,	-.5913-00)
.1292+01	(.1103+00 ,	.1074+01)	(-.1416+01 ,	-.6136-00)
.1326+01	(.1103+00 ,	.1090+01)	(-.1401+01 ,	-.6355-00)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Continued

(b) k = 2, d = 2

.1361+01	(.1098+00 ,	.1104+01)	(-.1386+01 ,	-.6571-00)
.1396+01	(.1086+00 ,	.1118+01)	(-.1371+01 ,	-.6784-00)
.1431+01	(.1068+00 ,	.1132+01)	(-.1355+01 ,	-.6995-00)
.1466+01	(.1044+00 ,	.1145+01)	(-.1340+01 ,	-.7202-00)
.1501+01	(.1014+00 ,	.1157+01)	(-.1325+01 ,	-.7405-00)
.1536+01	(.9779-01 ,	.1169+01)	(-.1309+01 ,	-.7606-00)
.1571+01	(.9362-01 ,	.1180+01)	(-.1294+01 ,	-.7804-00)
.1606+01	(.8888-01 ,	.1191+01)	(-.1278+01 ,	-.7998-00)
.1641+01	(.8358-01 ,	.1202+01)	(-.1262+01 ,	-.8189-00)
.1676+01	(.7773-01 ,	.1212+01)	(-.1246+01 ,	-.8377-00)
.1710+01	(.7135-01 ,	.1222+01)	(-.1230+01 ,	-.8561-00)
.1745+01	(.6444-01 ,	.1231+01)	(-.1214+01 ,	-.8743-00)
.1780+01	(.5701-01 ,	.1240+01)	(-.1198+01 ,	-.8921-00)
.1815+01	(.4909-01 ,	.1248+01)	(-.1181+01 ,	-.9095-00)
.1850+01	(.4068-01 ,	.1257+01)	(-.1165+01 ,	-.9267-00)
.1885+01	(.3180-01 ,	.1264+01)	(-.1148+01 ,	-.9435-00)
.1920+01	(.2246-01 ,	.1272+01)	(-.1132+01 ,	-.9600-00)
.1955+01	(.1268-01 ,	.1279+01)	(-.1115+01 ,	-.9761-00)
.1990+01	(.2462-02 ,	.1286+01)	(-.1098+01 ,	-.9919-00)
.2025+01	(-.8176-02 ,	.1292+01)	(-.1081+01 ,	-.1007+01)
.2059+01	(-.1922-01 ,	.1298+01)	(-.1064+01 ,	-.1023+01)
.2094+01	(-.3066-01 ,	.1304+01)	(-.1046+01 ,	-.1037+01)
.2129+01	(-.4248-01 ,	.1309+01)	(-.1029+01 ,	-.1052+01)
.2164+01	(-.5466-01 ,	.1314+01)	(-.1011+01 ,	-.1066+01)
.2199+01	(-.6721-01 ,	.1319+01)	(-.9937-00 ,	-.1080+01)
.2234+01	(-.8010-01 ,	.1323+01)	(-.9760-00 ,	-.1093+01)
.2269+01	(-.9332-01 ,	.1327+01)	(-.9582-00 ,	-.1106+01)
.2304+01	(-.1069+00 ,	.1331+01)	(-.9404-00 ,	-.1119+01)
.2339+01	(-.1207+00 ,	.1335+01)	(-.9224-00 ,	-.1132+01)
.2374+01	(-.1348-00 ,	.1338+01)	(-.9044-00 ,	-.1144+01)
.2409+01	(-.1493-00 ,	.1341+01)	(-.8863-00 ,	-.1155+01)
.2443+01	(-.1640-00 ,	.1343+01)	(-.8681-00 ,	-.1167+01)
.2478+01	(-.1789-00 ,	.1345+01)	(-.8498-00 ,	-.1178+01)
.2513+01	(-.1942-00 ,	.1347+01)	(-.8315-00 ,	-.1189+01)
.2548+01	(-.2096-00 ,	.1348+01)	(-.8132-00 ,	-.1199+01)
.2583+01	(-.2253-00 ,	.1350+01)	(-.7948-00 ,	-.1209+01)
.2618+01	(-.2412-00 ,	.1350+01)	(-.7764-00 ,	-.1219+01)
.2653+01	(-.2574-00 ,	.1351+01)	(-.7579-00 ,	-.1228+01)
.2688+01	(-.2737-00 ,	.1351+01)	(-.7395-00 ,	-.1237+01)
.2723+01	(-.2902-00 ,	.1351+01)	(-.7210-00 ,	-.1246+01)
.2758+01	(-.3069-00 ,	.1350+01)	(-.7024-00 ,	-.1254+01)
.2793+01	(-.3238-00 ,	.1350+01)	(-.6839-00 ,	-.1262+01)
.2827+01	(-.3408-00 ,	.1348+01)	(-.6654-00 ,	-.1270+01)
.2862+01	(-.3580-00 ,	.1347+01)	(-.6469-00 ,	-.1277+01)
.2897+01	(-.3754-00 ,	.1345+01)	(-.6284-00 ,	-.1284+01)
.2932+01	(-.3928-00 ,	.1343+01)	(-.6099-00 ,	-.1291+01)
.2967+01	(-.4104-00 ,	.1340+01)	(-.5915-00 ,	-.1297+01)
.3002+01	(-.4281-00 ,	.1338+01)	(-.5731-00 ,	-.1303+01)
.3037+01	(-.4460-00 ,	.1334+01)	(-.5547-00 ,	-.1308+01)
.3072+01	(-.4639-00 ,	.1331+01)	(-.5364-00 ,	-.1313+01)
.3107+01	(-.4819-00 ,	.1327+01)	(-.5182-00 ,	-.1318+01)
.3142+01	(-.5000-00 ,	.1323+01)	(-.5000-00 ,	-.1323+01)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Continued

(c) k = 3, d = 3

```
--- BASHFORTH-MOULTON  K=3,  D=3  ---

ALPHA( 3) =  .1000000000000000+01          BETA( 3) =  .0000000000000000
ALPHA( 2) = -.1000000000000000+01          BETA( 2) =  .1916666666666666+01
ALPHA( 1) =  .0000000000000000             BETA( 1) □ -.1333333333333332+01
ALPHA( 0) =  .0000000000000000             BETA( 0) =  .4166666666666666+00

GAMMA( 3) □  .1000000000000000+01          DELTA( 3) =  .4166666666666666+00
GAMMA( 2) = -.1000000000000000+01          DELTA( 2) =  .6666666666666665+00
GAMMA( 1) =  .0000000000000000             DELTA( 1) = -.8333333333333331-01
GAMMA( 0) =  .0000000000000000             DELTA( 0) =  .0000000000000000
```

THETA	HBAR1	HBAR2
.0000	(.0000 , -.0000)	(-.2400+01 , -.0000)
.3491-01	(-.6141-07 , .3491-01)	(-.2400+01 , .4247-04)
.6981-01	(-.9610-06 , .6981-01)	(-.2400+01 , .3383-03)
.1047+00	(-.4684-05 , .1047+00)	(-.2400+01 , .1134-02)
.1396-00	(-.1401-04 , .1396-00)	(-.2399+01 , .2659-02)
.1745-00	(-.3174-04 , .1746-00)	(-.2399+01 , .5124-02)
.2094-00	(-.5964-04 , .2095-00)	(-.2397+01 , .8705-02)
.2443-00	(-.9719-04 , .2445-00)	(-.2395+01 , .1354-01)
.2793-00	(-.1400-03 , .2795-00)	(-.2392+01 , .1972-01)
.3142-00	(-.1783-03 , .3147-00)	(-.2387+01 , .2728-01)
.3491-00	(-.1949-03 , .3499-00)	(-.2380+01 , .3618-01)
.3840-00	(-.1633-03 , .3853-00)	(-.2371+01 , .4632-01)
.4189-00	(-.4534-04 , .4209-00)	(-.2359+01 , .5750-01)
.4538-00	(.2105-03 , .4567-00)	(-.2345+01 , .6946-01)
.4887-00	(.6720-03 , .4928-00)	(-.2327+01 , .8185-01)
.5236-00	(.1425-02 , .5291-00)	(-.2306+01 , .9422-01)
.5585-00	(.2574-02 , .5658-00)	(-.2281+01 , .1061+00)
.5934-00	(.4247-02 , .6027-00)	(-.2253+01 , .1169+00)
.6283-00	(.6591-02 , .6400-00)	(-.2222+01 , .1261-00)
.6632-00	(.9770-02 , .6775-00)	(-.2187+01 , .1332-00)
.6981-00	(.1396-01 , .7152-00)	(-.2149+01 , .1375-00)
.7330-00	(.1935-01 , .7530-00)	(-.2109+01 , .1388-00)
.7679-00	(.2611-01 , .7905-00)	(-.2067+01 , .1366-00)
.8029-00	(.3439-01 , .8277-00)	(-.2023+01 , .1308-00)
.8378-00	(.4428-01 , .8642-00)	(-.1979+01 , .1214+00)
.8727-00	(.5581-01 , .8997-00)	(-.1935+01 , .1084+00)
.9076-00	(.6894-01 , .9337-00)	(-.1892+01 , .9211-01)
.9425-00	(.8352-01 , .9661-00)	(-.1850+01 , .7292-01)
.9774-00	(.9932-01 , .9963-00)	(-.1809+01 , .5123-01)
.1012+01	(.1161+00 , .1024+01)	(-.1769+01 , .2748-01)
.1047+01	(.1334-00 , .1050+01)	(-.1732+01 , .2118-02)
.1082+01	(.1509-00 , .1073+01)	(-.1696+01 , -.2445-01)
.1117+01	(.1683-00 , .1093+01)	(-.1662+01 , -.5186-01)
.1152+01	(.1852-00 , .1111+01)	(-.1629+01 , -.7980-01)
.1187+01	(.2014-00 , .1126+01)	(-.1599+01 , -.1080+00)
.1222+01	(.2165-00 , .1139+01)	(-.1569+01 , -.1364-00)
.1257+01	(.2304-00 , .1150+01)	(-.1541+01 , -.1646-00)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Continued

(c) k = 3, d = 3

.1292+01	(.2429-00 ,	.1159+01)	(-.1514+01 ,	-.1927-00)
.1326+01	(.2539-00 ,	.1167+01)	(-.1487+01 ,	-.2204-00)
.1361+01	(.2633-00 ,	.1173+01)	(-.1462+01 ,	-.2478-00)
.1396+01	(.2712-00 ,	.1178+01)	(-.1437+01 ,	-.2749-00)
.1431+01	(.2775-00 ,	.1182+01)	(-.1413+01 ,	-.3015-00)
.1466+01	(.2823-00 ,	.1185+01)	(-.1390+01 ,	-.3276-00)
.1501+01	(.2855-00 ,	.1187+01)	(-.1367+01 ,	-.3533-00)
.1536+01	(.2872-00 ,	.1188+01)	(-.1344+01 ,	-.3784-00)
.1571+01	(.2876-00 ,	.1189+01)	(-.1322+01 ,	-.4031-00)
.1606+01	(.2865-00 ,	.1190+01)	(-.1300+01 ,	-.4273-00)
.1641+01	(.2842-00 ,	.1190+01)	(-.1278+01 ,	-.4511-00)
.1676+01	(.2807-00 ,	.1190+01)	(-.1257+01 ,	-.4743-00)
.1710+01	(.2760-00 ,	.1190+01)	(-.1236+01 ,	-.4971-00)
.1745+01	(.2703-00 ,	.1190+01)	(-.1215+01 ,	-.5194-00)
.1780+01	(.2635-00 ,	.1189+01)	(-.1194+01 ,	-.5412-00)
.1815+01	(.2558-00 ,	.1189+01)	(-.1173+01 ,	-.5626-00)
.1850+01	(.2472-00 ,	.1188+01)	(-.1152+01 ,	-.5835-00)
.1885+01	(.2377-00 ,	.1187+01)	(-.1131+01 ,	-.6040-00)
.1920+01	(.2275-00 ,	.1186+01)	(-.1110+01 ,	-.6240-00)
.1955+01	(.2165-00 ,	.1185+01)	(-.1090+01 ,	-.6436-00)
.1990+01	(.2049-00 ,	.1185+01)	(-.1069+01 ,	-.6628-00)
.2025+01	(.1926-00 ,	.1184+01)	(-.1049+01 ,	-.6815-00)
.2059+01	(.1797-00 ,	.1182+01)	(-.1028+01 ,	-.6998-00)
.2094+01	(.1663-00 ,	.1181+01)	(-.1007+01 ,	-.7177-00)
.2129+01	(.1523-00 ,	.1180+01)	(-.9866-00 ,	-.7351-00)
.2164+01	(.1379-00 ,	.1179+01)	(-.9659-00 ,	-.7522-00)
.2199+01	(.1229+00 ,	.1178+01)	(-.9453-00 ,	-.7689-00)
.2234+01	(.1076+00 ,	.1176+01)	(-.9246-00 ,	-.7851-00)
.2269+01	(.9183-01 ,	.1175+01)	(-.9039-00 ,	-.8010-00)
.2304+01	(.7568-01 ,	.1173+01)	(-.8831-00 ,	-.8164-00)
.2339+01	(.5918-01 ,	.1171+01)	(-.8623-00 ,	-.8315-00)
.2374+01	(.4234-01 ,	.1169+01)	(-.8415-00 ,	-.8462-00)
.2409+01	(.2519-01 ,	.1167+01)	(-.8207-00 ,	-.8605-00)
.2443+01	(.7740-02 ,	.1165+01)	(-.7999-00 ,	-.8744-00)
.2478+01	(-.9988-02 ,	.1163+01)	(-.7790-00 ,	-.8880-00)
.2513+01	(-.2798-01 ,	.1160+01)	(-.7581-00 ,	-.9012-00)
.2548+01	(-.4621-01 ,	.1158+01)	(-.7372-00 ,	-.9140-00)
.2583+01	(-.6468-01 ,	.1155+01)	(-.7163-00 ,	-.9264-00)
.2618+01	(-.8336-01 ,	.1152+01)	(-.6953-00 ,	-.9385-00)
.2653+01	(-.1022+00 ,	.1149+01)	(-.6744-00 ,	-.9502-00)
.2688+01	(-.1213+00 ,	.1145+01)	(-.6534-00 ,	-.9616-00)
.2723+01	(-.1406-00 ,	.1142+01)	(-.6324-00 ,	-.9726-00)
.2758+01	(-.1600-00 ,	.1138+01)	(-.6114-00 ,	-.9833-00)
.2793+01	(-.1796-00 ,	.1134+01)	(-.5905-00 ,	-.9936-00)
.2827+01	(-.1993-00 ,	.1129+01)	(-.5695-00 ,	-.1004+01)
.2862+01	(-.2192-00 ,	.1125+01)	(-.5485-00 ,	-.1013+01)
.2897+01	(-.2391-00 ,	.1120+01)	(-.5276-00 ,	-.1022+01)
.2932+01	(-.2592-00 ,	.1115+01)	(-.5066-00 ,	-.1031+01)
.2967+01	(-.2794-00 ,	.1110+01)	(-.4857-00 ,	-.1040+01)
.3002+01	(-.2998-00 ,	.1104+01)	(-.4649-00 ,	-.1048+01)
.3037+01	(-.3202-00 ,	.1098+01)	(-.4440-00 ,	-.1056+01)
.3072+01	(-.3406-00 ,	.1092+01)	(-.4232-00 ,	-.1064+01)
.3107+01	(-.3612-00 ,	.1085+01)	(-.4025-00 ,	-.1071+01)
.3142+01	(-.3818-00 ,	.1079+01)	(-.3818-00 ,	-.1079+01)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON PREDICTOR-CORRECTOR PAIRS - Continued

(d) k = 4, d = 4

--- BASHFORTH-MOULTON K=4, D=4 ---

```
ALPHA( 4) =  .1000000000000000+01          BETA( 4) =  .0000000000000000
ALPHA( 3) = -.1000000000000000+01          BETA( 3) =  .2291666666666666+01
ALPHA( 2) =  .0000000000000000             BETA( 2) = -.2458333333333332+01
ALPHA( 1) =  .0000000000000000             BETA( 1) =  .1541666666666666+01
ALPHA( 0) =  .0000000000000000             BETA( 0) = -.3750000000000000+00

GAMMA( 4) =  .1000000000000000+01          DELTA( 4) =  .3750000000000000+00
GAMMA( 3) ■ -.1000000000000000+01          DELTA( 3) =  .7916666666666665+00
GAMMA( 2) =  .0000000000000000             DELTA( 2) = -.2083333333333332+00
GAMMA( 1) =  .0000000000000000             DELTA( 1) =  .4166666666666666-01
GAMMA( 0) =  .0000000000000000             DELTA( 0) =  .0000000000000000
```

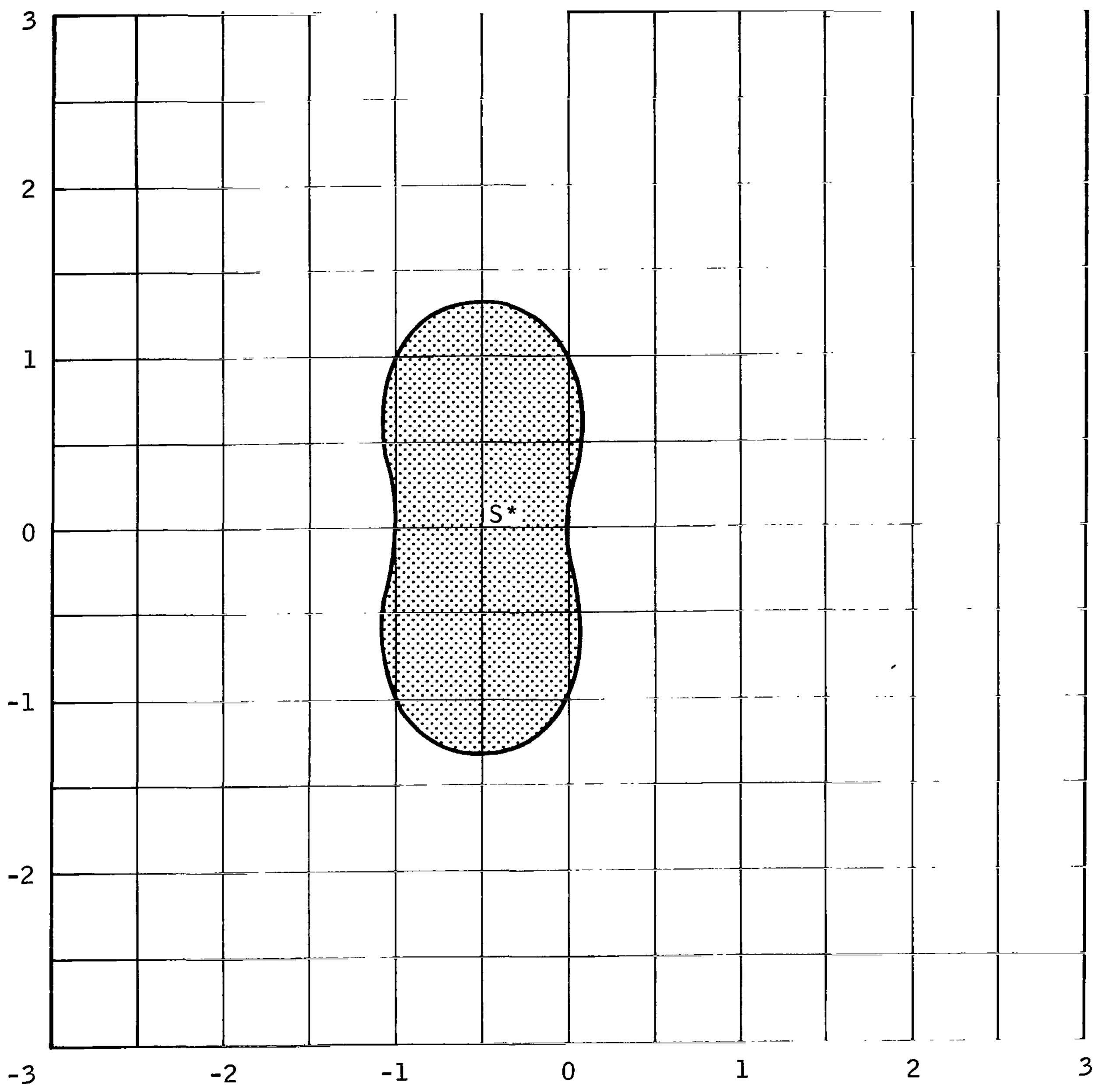
THETA	HBAR1		HBAR2	
.0000	(.0000 ,	-.0000)	(-.2667+01 ,	-.0000)
.3491-01	(-.2915-09 ,	.3491-01)	(-.2667+01 ,	.9708-07)
.6981-01	(-.1854-07 ,	.6981-01)	(-.2667+01 ,	.3098-05)
.1047+00	(-.2089-06 ,	.1047+00)	(-.2667+01 ,	.2341-04)
.1396-00	(-.1157-05 ,	.1396-00)	(-.2667+01 ,	.9800-04)
.1745-00	(-.4327-05 ,	.1745-00)	(-.2668+01 ,	.2966-03)
.2094-00	(-.1262-04 ,	.2094-00)	(-.2668+01 ,	.7306-03)
.2443-00	(-.3092-04 ,	.2443-00)	(-.2670+01 ,	.1561-02)
.2793-00	(-.6665-04 ,	.2792-00)	(-.2672+01 ,	.3003-02)
.3142-00	(-.1300-03 ,	.3142-00)	(-.2675+01 ,	.5333-02)
.3491-00	(-.2343-03 ,	.3491-00)	(-.2678+01 ,	.8888-02)
.3840-00	(-.3952-03 ,	.3841-00)	(-.2683+01 ,	.1407-01)
.4189-00	(-.6307-03 ,	.4191-00)	(-.2688+01 ,	.2133-01)
.4538-00	(-.9594-03 ,	.4542-00)	(-.2694+01 ,	.3120-01)
.4887-00	(-.1399-02 ,	.4895-00)	(-.2700+01 ,	.4422-01)
.5236-00	(-.1964-02 ,	.5249-00)	(-.2706+01 ,	.6101-01)
.5585-00	(-.2663-02 ,	.5607-00)	(-.2712+01 ,	.8215-01)
.5934-00	(-.3493-02 ,	.5969-00)	(-.2716+01 ,	.1082+00)
.6283-00	(-.4437-02 ,	.6336-00)	(-.2718+01 ,	.1397-00)
.6632-00	(-.5454-02 ,	.6709-00)	(-.2716+01 ,	.1769-00)
.6981-00	(-.6474-02 ,	.7091-00)	(-.2709+01 ,	.2200-00)
.7330-00	(-.7382-02 ,	.7484-00)	(-.2696+01 ,	.2687-00)
.7679-00	(-.8000-02 ,	.7890-00)	(-.2673+01 ,	.3221-00)
.8029-00	(-.8068-02 ,	.8312-00)	(-.2640+01 ,	.3790-00)
.8378-00	(-.7204-02 ,	.8753-00)	(-.2595+01 ,	.4371-00)
.8727-00	(-.4866-02 ,	.9217-00)	(-.2536+01 ,	.4936-00)
.9076-00	(-.2893-03 ,	.9704-00)	(-.2463+01 ,	.5449-00)
.9425-00	(.7564-02 ,	.1022+01)	(-.2378+01 ,	.5871-00)
.9774-00	(.2004-01 ,	.1075+01)	(-.2281+01 ,	.6166-00)
.1012+01	(.3871-01 ,	.1131+01)	(-.2178+01 ,	.6303-00)
.1047+01	(.6515-01 ,	.1186+01)	(-.2073+01 ,	.6271-00)
.1082+01	(.1004+00 ,	.1239+01)	(-.1970+01 ,	.6076-00)
.1117+01	(.1443-00 ,	.1286+01)	(-.1876+01 ,	.5744-00)
.1152+01	(.1954-00 ,	.1325+01)	(-.1790+01 ,	.5315-00)
.1187+01	(.2506-00 ,	.1352+01)	(-.1716+01 ,	.4828-00)

TABLE BII. - LINE PRINTOUTS OF BASHFORTH-MOULTON

PREDICTOR-CORRECTOR PAIRS - Concluded

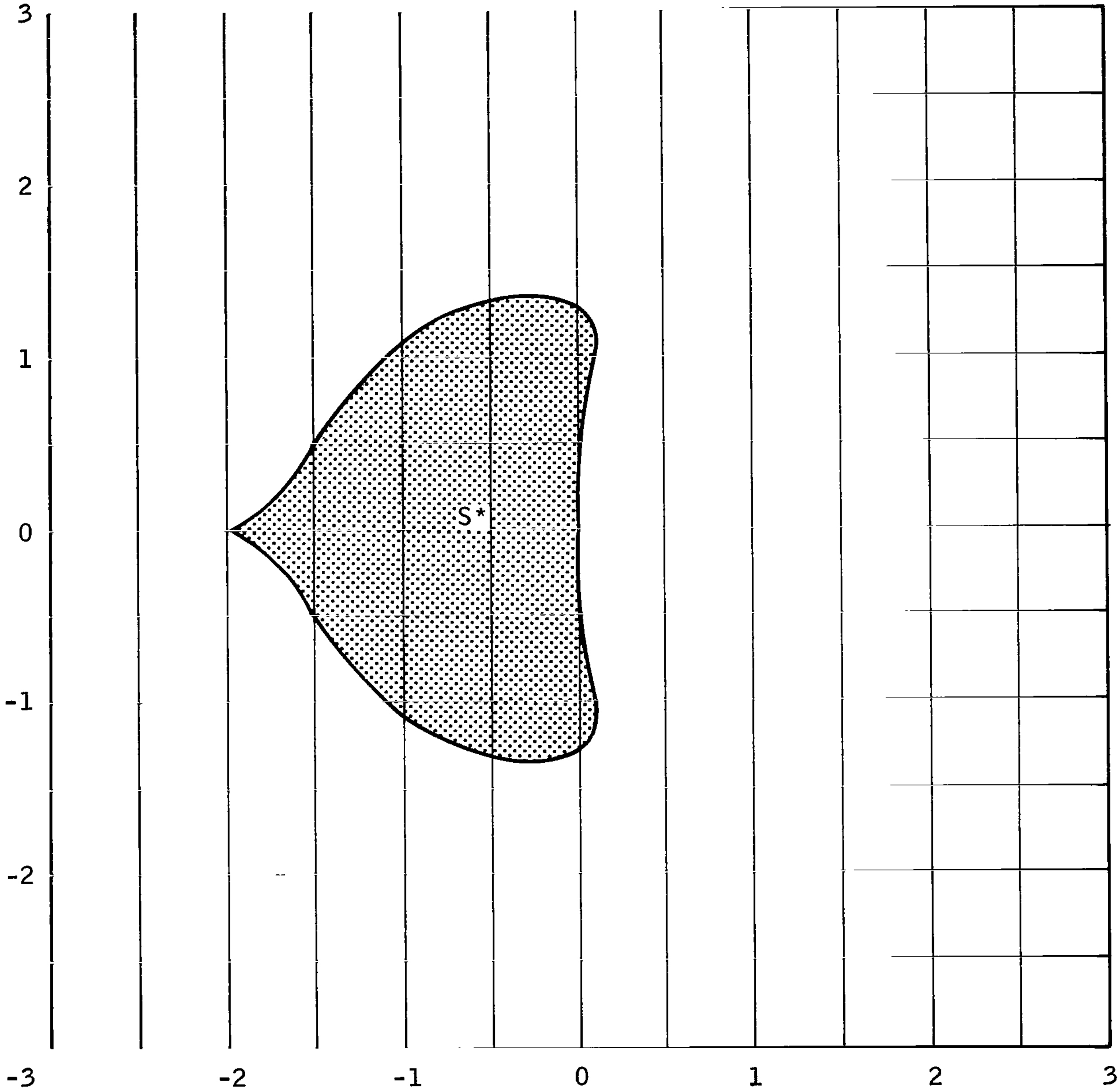
(d) k = 4, d = 4

.1222+01	(.3068-00 ,	.1369+01)	(-.1651+01 ,	.4313-00)
.1257+01	(.3608-00 ,	.1374+01)	(-.1594+01 ,	.3795-00)
.1292+01	(.4103-00 ,	.1370+01)	(-.1545+01 ,	.3285-00)
.1326+01	(.4538-00 ,	.1358+01)	(-.1501+01 ,	.2792-00)
.1361+01	(.4905-00 ,	.1341+01)	(-.1461+01 ,	.2318-00)
.1396+01	(.5203-00 ,	.1321+01)	(-.1425+01 ,	.1864-00)
.1431+01	(.5435-00 ,	.1298+01)	(-.1391+01 ,	.1430-00)
.1466+01	(.5605-00 ,	.1274+01)	(-.1360+01 ,	.1015+00)
.1501+01	(.5719-00 ,	.1249+01)	(-.1330+01 ,	.6182-01)
.1536+01	(.5783-00 ,	.1225+01)	(-.1302+01 ,	.2381-01)
.1571+01	(.5805-00 ,	.1202+01)	(-.1275+01 ,	-.1265-01)
.1606+01	(.5790-00 ,	.1181+01)	(-.1250+01 ,	-.4767-01)
.1641+01	(.5742-00 ,	.1160+01)	(-.1225+01 ,	-.8135-01)
.1676+01	(.5668-00 ,	.1141+01)	(-.1200+01 ,	-.1138+00)
.1710+01	(.5571-00 ,	.1124+01)	(-.1176+01 ,	-.1450-00)
.1745+01	(.5454-00 ,	.1107+01)	(-.1153+01 ,	-.1752-00)
.1780+01	(.5321-00 ,	.1093+01)	(-.1130+01 ,	-.2043-00)
.1815+01	(.5174-00 ,	.1079+01)	(-.1107+01 ,	-.2325-00)
.1850+01	(.5016-00 ,	.1067+01)	(-.1085+01 ,	-.2598-00)
.1885+01	(.4848-00 ,	.1055+01)	(-.1063+01 ,	-.2862-00)
.1920+01	(.4672-00 ,	.1045+01)	(-.1041+01 ,	-.3118-00)
.1955+01	(.4489-00 ,	.1036+01)	(-.1019+01 ,	-.3365-00)
.1990+01	(.4299-00 ,	.1027+01)	(-.9973-00 ,	-.3606-00)
.2025+01	(.4106-00 ,	.1019+01)	(-.9757-00 ,	-.3839-00)
.2059+01	(.3907-00 ,	.1012+01)	(-.9542-00 ,	-.4065-00)
.2094+01	(.3706-00 ,	.1005+01)	(-.9327-00 ,	-.4284-00)
.2129+01	(.3501-00 ,	.9991-00)	(-.9113-00 ,	-.4497-00)
.2164+01	(.3294-00 ,	.9934-00)	(-.8899-00 ,	-.4704-00)
.2199+01	(.3085-00 ,	.9881-00)	(-.8686-00 ,	-.4905-00)
.2234+01	(.2873-00 ,	.9831-00)	(-.8472-00 ,	-.5099-00)
.2269+01	(.2661-00 ,	.9783-00)	(-.8259-00 ,	-.5288-00)
.2304+01	(.2447-00 ,	.9738-00)	(-.8045-00 ,	-.5472-00)
.2339+01	(.2231-00 ,	.9694-00)	(-.7831-00 ,	-.5650-00)
.2374+01	(.2015-00 ,	.9652-00)	(-.7617-00 ,	-.5823-00)
.2409+01	(.1798-00 ,	.9610-00)	(-.7403-00 ,	-.5990-00)
.2443+01	(.1580-00 ,	.9569-00)	(-.7189-00 ,	-.6153-00)
.2478+01	(.1362-00 ,	.9528-00)	(-.6974-00 ,	-.6310-00)
.2513+01	(.1142+00 ,	.9487-00)	(-.6759-00 ,	-.6463-00)
.2548+01	(.9230-01 ,	.9445-00)	(-.6544-00 ,	-.6611-00)
.2583+01	(.7031-01 ,	.9403-00)	(-.6329-00 ,	-.6755-00)
.2618+01	(.4828-01 ,	.9360-00)	(-.6113-00 ,	-.6893-00)
.2653+01	(.2623-01 ,	.9316-00)	(-.5896-00 ,	-.7028-00)
.2688+01	(.4156-02 ,	.9271-00)	(-.5680-00 ,	-.7158-00)
.2723+01	(-.1794-01 ,	.9224-00)	(-.5463-00 ,	-.7283-00)
.2758+01	(-.4005-01 ,	.9175-00)	(-.5245-00 ,	-.7405-00)
.2793+01	(-.6217-01 ,	.9124-00)	(-.5027-00 ,	-.7522-00)
.2827+01	(-.8430-01 ,	.9072-00)	(-.4809-00 ,	-.7635-00)
.2862+01	(-.1064+00 ,	.9017-00)	(-.4591-00 ,	-.7744-00)
.2897+01	(-.1286-00 ,	.8960-00)	(-.4372-00 ,	-.7849-00)
.2932+01	(-.1507-00 ,	.8900-00)	(-.4153-00 ,	-.7951-00)
.2967+01	(-.1728-00 ,	.8838-00)	(-.3934-00 ,	-.8049-00)
.3002+01	(-.1950-00 ,	.8773-00)	(-.3714-00 ,	-.8143-00)
.3037+01	(-.2171-00 ,	.8705-00)	(-.3494-00 ,	-.8233-00)
.3072+01	(-.2392-00 ,	.8634-00)	(-.3274-00 ,	-.8320-00)
.3107+01	(-.2613-00 ,	.8561-00)	(-.3054-00 ,	-.8403-00)
.3142+01	(-.2833-00 ,	.8484-00)	(-.2833-00 ,	-.8484-00)



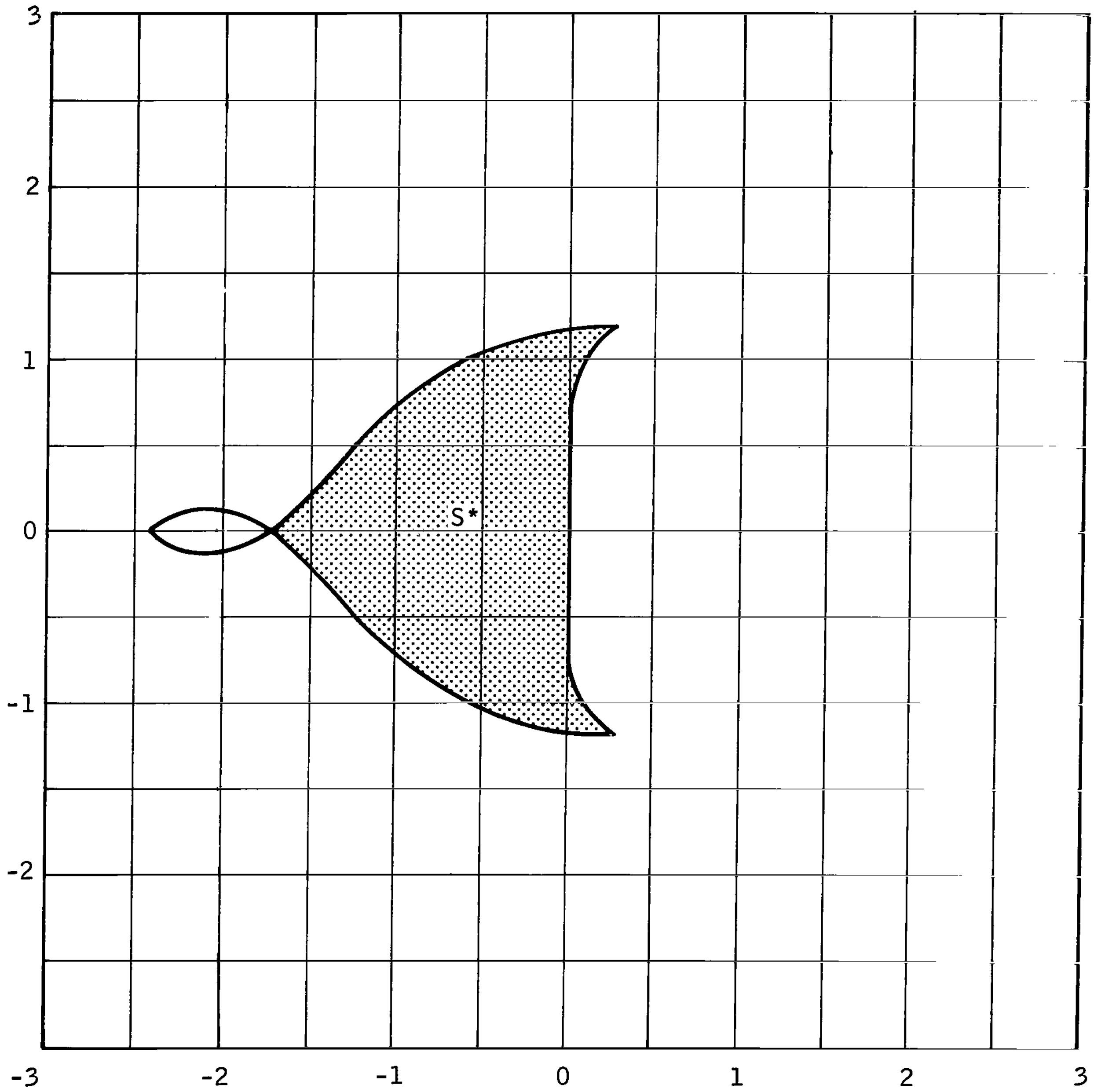
(a) k = 1, d = 1

Figure B1. - FORTRAN IV subroutine STBL2 plots of Bashforth-Moulton predictor-corrector pairs.



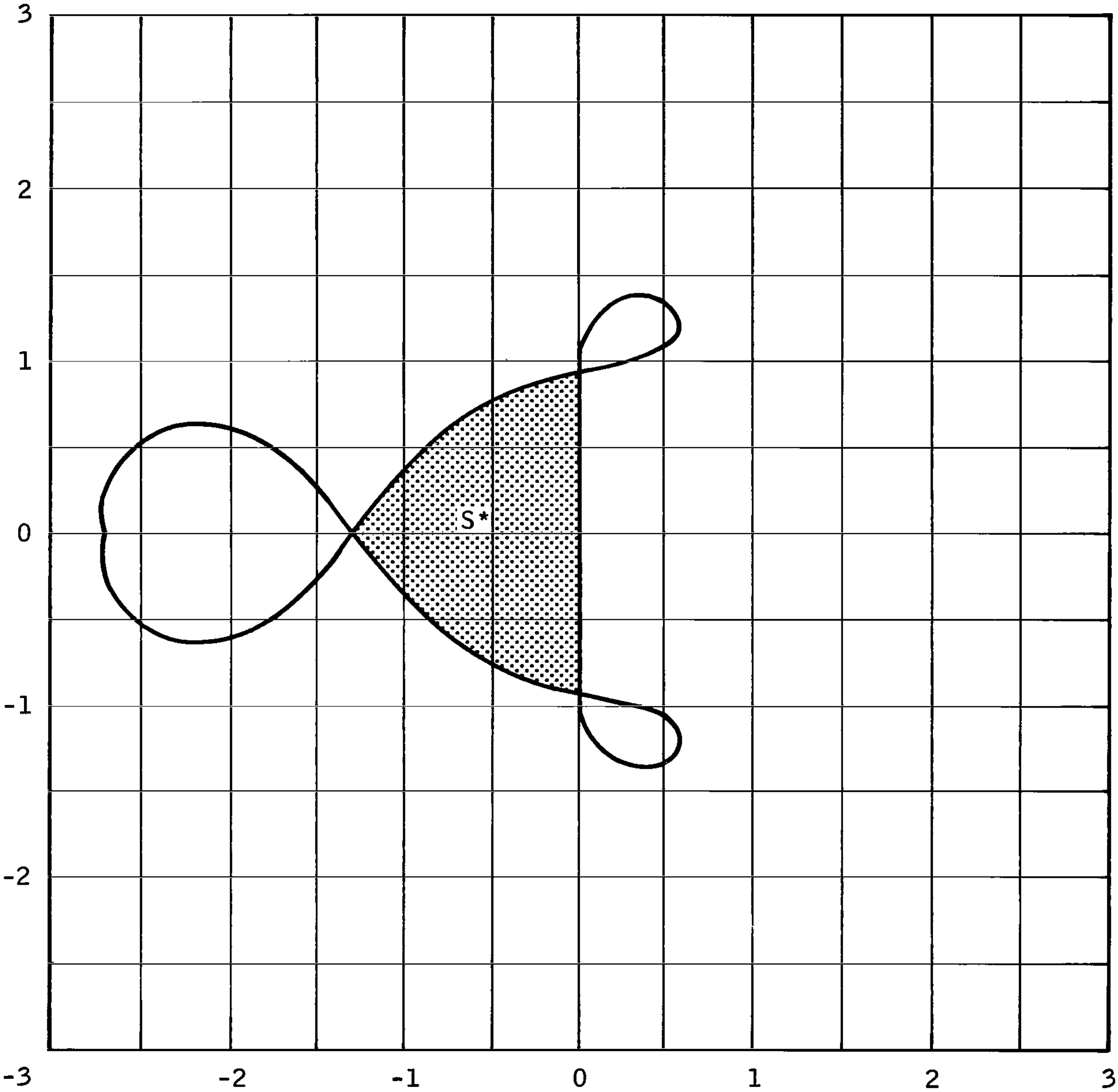
(b) k = 2, d = 2

Figure B1. - Continued.



(c) k = 3, d = 3

Figure B1. - Continued.



(d) k = 4, d = 4

Figure B1. - Concluded.

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